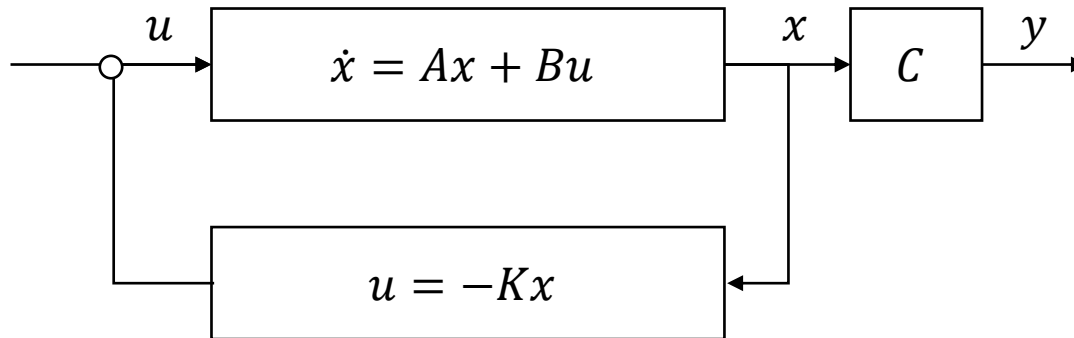


Modern Control Theory

Model Predictive Control (MPC)



Standard MPC performance index

The MPC controller minimize the standard performance index at time k :

$$\sum_{i=0}^{N_p-1} z^T(k+i)\Omega(i)z(k+i) = \sum_{i=N_m}^{N_p-1} z_1^T(k+i)z_1(k+i) + \sum_{i=0}^{N_c-1} z_2^T(k+i)z_2(k+i)$$

where

z_1 reflects tracking error and z_2 reflects the control action,

$z_j(k+i) = z_j(k+i|k)$ is prediction of $z_j(k+i)$ at time k .

N_p prediction horizon

N_m minimum control horizon

N_c control horizon

Control horizon performance index

Choose

$$\tilde{\Omega} = \text{diag}(\tilde{\Omega}_1, \tilde{\Omega}_2) = \text{diag}(I_{N_p}, I_{N_c}, 0_{N_p-N_c}), \quad \tilde{\Omega}_2 = \text{diag}(I_{N_c}, 0_{N_p-N_c})$$

Then

$$\tilde{z}_1^T(k) \tilde{\Omega}_1 \tilde{z}_1(k) = \sum_{i=0}^{N_p-1} z_1^T(k+i) z_1(k+i)$$

$$\tilde{z}_2^T(k) \tilde{\Omega}_2 \tilde{z}_2(k) = \sum_{i=0}^{N_c-1} z_2^T(k+i) z_2(k+i)$$

s.t.

$$\tilde{z}^T(k) \tilde{\Omega} \tilde{z}(k) = \tilde{z}_1^T(k) \tilde{\Omega}_1 \tilde{z}_1(k) + \tilde{z}_2^T(k) \tilde{\Omega}_2 \tilde{z}_2(k)$$

$$\sum_{i=0}^{N_p-1} z^T(k+i) \Omega(i) z(k+i) = \sum_{i=0}^{N_p-1} z_1^T(k+i) z_1(k+i) + \sum_{i=0}^{N_c-1} z_2^T(k+i) z_2(k+i)$$

LQPC performance index

$$z(k) = \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} = \begin{bmatrix} C_1 x(k) \\ D_{12} v(k) \end{bmatrix}$$

Choose $\tilde{\Omega} = I$

Let $Q = C_1^T C_1, R = D_{12}^T D_{12}$

Then

$$\tilde{z}^T(k) \tilde{\Omega} \tilde{z}(k) = \tilde{z}_1^T(k) \tilde{z}_1(k) + \tilde{z}_2^T(k) \tilde{z}_2(k)$$

$$\begin{aligned} \sum_{i=0}^{N_p-1} z^T(k+i) \Omega(i) z(k+i) &= \sum_{i=0}^{N_p-1} z_1^T(k+i) z_1(k+i) + z_2^T(k+i) z_2(k+i) \\ &= \sum_{i=0}^{N_p-1} x^T(k+i) Q x(k+i) + v^T(k+i) R v(k+i) \end{aligned}$$

GPC performance index

$$z(k) = \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} = \begin{bmatrix} C_1 x(k) - r(k) \\ D_{12} \Delta u(k) \end{bmatrix}$$

Choose $\tilde{\Omega} = \text{diag}(\tilde{\Omega}_1, \tilde{\Omega}_2) = \text{diag}(\tilde{\Omega}_1, I)$, $\tilde{\Omega}_1 = \text{diag}(0_{N_m}, I_{N_p - N_m})$

Then

$$\tilde{z}^T(k) \tilde{\Omega} \tilde{z}(k) = \tilde{z}_1^T(k) \tilde{\Omega}_1 \tilde{z}_1(k) + \tilde{z}_2^T(k) \tilde{z}_2(k)$$

$$\begin{aligned} \sum_{i=0}^{N_p-1} z^T(k+i) \Omega(i) z(k+i) &= \sum_{i=N_m}^{N_p-1} (\eta(k+i) - r(k+i))^T (\eta(k+i) - r(k+i)) \\ &\quad + \sum_{i=0}^{N_p-1} \Delta u^T(k+i) R \Delta u(k+i) \end{aligned}$$

Prediction of controlled variable

Let us consider

$$x(k+1) = \Phi x(k) + \Gamma_2 v(k)$$

$$z_1(k) = \eta(k) - r(k)$$

$$z_2(k) = D_{12}v(k).$$

where $\eta(k) = C_1 x(k)$

Let

$$\tilde{x}(k) = \begin{bmatrix} x(k) \\ \vdots \\ x(k + N_p - 1) \end{bmatrix}, \quad \tilde{r}(k) = \begin{bmatrix} r(k) \\ \vdots \\ r(k + N_p - 1) \end{bmatrix}.$$

Prediction of controlled variable

Also let

$$\tilde{v}(k) = \begin{bmatrix} v(k) \\ \vdots \\ v(k + N_c - 1) \end{bmatrix}, \quad \tilde{v}_a(k) = \begin{bmatrix} u(k) \\ \vdots \\ v(k + N_c - 1) \\ \vdots \\ v(k + N_p - 1) \end{bmatrix}$$

and

$$\tilde{\eta}(k) = \begin{bmatrix} \eta(k) \\ \vdots \\ \eta(k + N_p - 1) \end{bmatrix} = \tilde{C}_1 \tilde{x}(k) = \text{diag}(C_1, \dots, C_1) \begin{bmatrix} x(k) \\ \vdots \\ x(k + N_p - 1) \end{bmatrix}.$$

Prediction of controlled variable

Consider state predictions

$$x(k+1) = \Phi x(k) + \Gamma_2 v(k)$$

$$x(k+2) = \Phi x(k+1) + \Gamma_2 v(k+1) = \Phi^2 x(k) + \Phi \Gamma_2 v(k) + \Gamma_2 v(k+1)$$

$$\vdots$$

$$x(k+N_p-1) = \Phi^{N_p-1} x(k) + \begin{bmatrix} \Phi^{N_p-2} \Gamma_2 & \Phi^{N_p-3} \Gamma_2 & \cdots & \Gamma_2 \end{bmatrix} \begin{bmatrix} v(k) \\ \vdots \\ v(k+N_p-2) \end{bmatrix}$$

Prediction of controlled variable

Thus

$$\tilde{x}(k) = \begin{bmatrix} I \\ \Phi \\ \vdots \\ \Phi^{N_p-1} \end{bmatrix} x(k) + \begin{bmatrix} 0_{n_x \times n_u(N_p-1)} & 0_{n_x \times n_u} \\ M & 0_{n_x(N_p-1) \times n_u} \end{bmatrix} \tilde{v}_a(k)$$

where

$$M = \begin{bmatrix} \Gamma_2 & & & \\ \Phi\Gamma_2 & \Gamma_2 & & \\ \vdots & & \ddots & \\ \Phi^{N_p-2}\Gamma_2 & \Phi^{N_p-3}\Gamma_2 & \cdots & \Gamma_2 \end{bmatrix} \in \mathbb{R}^{n_x(N_p-1) \times n_u(N_p-1)}$$

Prediction of controlled variable

Then

$$\tilde{\eta}(k) = \tilde{C}_1 \tilde{x}(k) = \Psi x(k) + \tilde{C}_1 \begin{bmatrix} 0_{n_x \times n_u (N_p - 1)} & 0_{n_x \times n_u} \\ M & 0_{n_x (N_p - 1) \times n_u} \end{bmatrix} \tilde{v}_a(k)$$

where

$$\Psi = \tilde{C}_1 \tilde{\Phi} = \tilde{C}_1 \begin{bmatrix} I \\ \Phi \\ \vdots \\ \Phi^{N_p - 1} \end{bmatrix}$$

Prediction with control from IO models

$$v(k) = u(k), u(k+i) = u(k+N_c-1) \text{ for } i = N_c, \dots, N_p-1$$

$$\Phi = \Phi_{io}, \Gamma_2 = \Gamma_{2,io}, C_1 = C_{1,io}, D_{12} = D_{12,io}$$

$$\tilde{v}_a(k) = \tilde{u}_a(k) = \begin{bmatrix} I_{n_u N_c} \\ \left[0_{n_u(N_p-N_c) \times (n_u N_c - 1)} \quad 1_{n_u(N_p-N_c) \times 1} \right] \end{bmatrix} \tilde{u}(k)$$

Thus

$$\tilde{\eta}(k) = \Psi x(k) + \Theta_{io} \tilde{u}(k)$$

where

$$\Theta_{io} = \tilde{C}_1 \begin{bmatrix} 0 & 0 \\ M & 0 \end{bmatrix} \begin{bmatrix} I_{n_u N_c} \\ \left[0_{n_u(N_p-N_c) \times (n_u N_c - 1)} \quad 1_{n_u(N_p-N_c) \times 1} \right] \end{bmatrix}$$

Prediction with incremental control from IIO model

$$v(k) = \Delta u(k), \Delta u(k+i) = 0 \text{ for } i = N_c, \dots, N_p - 1$$

$$\Phi = \Phi_{iio}, \Gamma_2 = \Gamma_{2,iio}, C_1 = C_{1,iio}, D_{12} = D_{12,iio}$$

$$\tilde{v}_a(k) = \Delta \tilde{u}_a(k) = \begin{bmatrix} I_{n_u N_c} \\ 0_{n_u(N_p - N_c)} \end{bmatrix} \Delta \tilde{u}(k)$$

Thus

$$\tilde{\eta}(k) = \Psi x(k) + \Theta_{iio} \Delta \tilde{u}(k)$$

where

$$\Theta_{iio} = \tilde{C}_1 \begin{bmatrix} 0 & 0 \\ M & 0 \end{bmatrix} \begin{bmatrix} I_{n_u N_c} \\ 0_{n_u(N_p - N_c)} \end{bmatrix}$$

Prediction with incremental control from IO model

$$v(k) = \Delta u(k), \Delta u(k+i) = 0 \text{ for } i = N_c, \dots, N_p - 1$$

$$\Phi = \Phi_{io}, \Gamma_2 = \Gamma_{2,io}, C_1 = C_{1,io}, D_{12} = D_{12,io}$$

$$x(k+1) = \Phi x(k) + \Gamma_2 u(k)$$

$$z_1(k) = \eta(k) - r(k).$$

$$\tilde{\eta}(k) = \tilde{C}_1 \tilde{x}(k)$$

Prediction with incremental control from IO model

Let

$$\tilde{u}(k) = \begin{bmatrix} u(k) \\ \vdots \\ u(k + N_c - 1) \end{bmatrix}, \quad \tilde{u}_a(k) = \begin{bmatrix} u(k) \\ \vdots \\ u(k + N_c - 1) \\ \vdots \\ u(k + N_p - 1) \end{bmatrix}$$

$$\Delta \tilde{u}(k) = \begin{bmatrix} \Delta u(k) \\ \vdots \\ \Delta u(k + N_c - 1) \end{bmatrix} = \tilde{u}(k) - \tilde{u}(k - 1)$$

$$\Delta \tilde{u}_a(k) = \begin{bmatrix} \Delta \tilde{u}(k) \\ 0_{n_u(N_p - N_c)} \end{bmatrix} = \tilde{u}_a(k) - \tilde{u}_a(k - 1)$$

Prediction with incremental control from IO model

Then

$$\tilde{x}(k) = \tilde{\Phi}x(k) + \begin{bmatrix} 0 & 0 \\ M & 0 \end{bmatrix} \tilde{u}_a(k)$$

and

$$\tilde{\eta}(k) = \Psi x(k) + \tilde{C}_1 \begin{bmatrix} 0 & 0 \\ M & 0 \end{bmatrix} \tilde{u}_a(k)$$

Prediction with incremental control from IO model

Note

$$\tilde{u}_a(k-1) = \begin{bmatrix} u(k-1) \\ \vdots \\ u(k+N_p-2) \end{bmatrix} = \begin{bmatrix} u(k-1) \\ 0_{n_u(N_p-1) \times 1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ I_{n_u(N_p-1)} & 0 \end{bmatrix} \tilde{u}_a(k)$$

and

$$\tilde{u}_a(k-1) = -\Delta \tilde{u}_a(k) + \tilde{u}_a(k)$$

$$\left(I_{n_u N_p} - \begin{bmatrix} 0 & 0 \\ I_{n_u(N_p-1)} & 0 \end{bmatrix} \right) \tilde{u}_a(k) = \Delta \tilde{u}_a(k) + \begin{bmatrix} u(k-1) \\ 0_{n_u(N_p-1) \times 1} \end{bmatrix}$$

Prediction with incremental control from IO model

Thus

$$\begin{aligned}\tilde{u}_a(k) &= \left(I_{n_u N_p} - \begin{bmatrix} 0 & 0 \\ I_{n_u(N_p-1)} & 0 \end{bmatrix} \right)^{-1} \left(\Delta \tilde{u}_a(k) + \begin{bmatrix} u(k-1) \\ 0 \end{bmatrix} \right) \\ &= \left(I_{n_u N_p} - \begin{bmatrix} 0 & 0 \\ I_{n_u(N_p-1)} & 0 \end{bmatrix} \right)^{-1} \Delta \tilde{u}_a(k) + \begin{bmatrix} u(k-1) \\ \vdots \\ u(k-1) \end{bmatrix}\end{aligned}$$

Prediction with incremental control from IO model

Note

$$\begin{aligned} & \tilde{C}_1 \begin{bmatrix} 0 & 0 \\ M & 0 \end{bmatrix} \tilde{u}_a(k) \\ &= \tilde{C}_1 \begin{bmatrix} 0 & 0 \\ M & 0 \end{bmatrix} \left(I_{n_u N_p} - \begin{bmatrix} 0 & 0 \\ I_{n_u(N_p-1)} & 0 \end{bmatrix} \right)^{-1} \left(\Delta \tilde{u}_a(k) + \begin{bmatrix} u(k-1) \\ 0_{n_u(N_p-1) \times 1} \end{bmatrix} \right) \\ &= \begin{bmatrix} \Theta & \times \end{bmatrix} \left(\Delta \tilde{u}_a(k) + \begin{bmatrix} u(k-1) \\ 0_{n_u(N_p-1) \times 1} \end{bmatrix} \right) \end{aligned}$$

where

Prediction with incremental control from IO model

$$\begin{bmatrix} \Theta & \times \end{bmatrix} = \tilde{C}_1 \begin{bmatrix} 0 & 0 \\ M & 0 \end{bmatrix} \left(I_{n_u N_p} - \begin{bmatrix} 0 & 0 \\ I_{n_u(N_p-1)} & 0 \end{bmatrix} \right)^{-1}$$

and

$$\Theta = \tilde{C}_1 \begin{bmatrix} 0 \\ \Gamma_2 \\ \vdots \\ \sum_{i=0}^{N_c-2} \Phi^i \Gamma_2 & \cdots & \Gamma_2 \\ \vdots & & \vdots \\ \sum_{i=0}^{N_p-2} \Phi^i \Gamma_2 & \cdots & \sum_{i=0}^{N_p-N_c-1} \Phi^i \Gamma_2 \end{bmatrix}$$

Prediction with incremental control from IO model

Note

$$\begin{bmatrix} \Theta & \times \end{bmatrix} \Delta \tilde{u}_a(k) = \Theta \Delta \tilde{u}(k)$$

and

$$\begin{bmatrix} \Theta & \times \end{bmatrix} \begin{bmatrix} u(k-1) \\ 0_{n_u(N_p-1) \times 1} \end{bmatrix} = \Upsilon u(k-1) = \tilde{C}_1 \begin{bmatrix} 0 \\ \Gamma_2 \\ \vdots \\ \sum_{i=0}^{N_c-2} \Phi^i \Gamma_2 \\ \vdots \\ \sum_{i=0}^{N_p-2} \Phi^i \Gamma_2 \end{bmatrix} u(k-1)$$

Prediction with incremental control from IO model

Then

$$\tilde{C}_1 \begin{bmatrix} 0 & 0 \\ M & 0 \end{bmatrix} \tilde{u}_a(k) = \Theta \left(\Delta \tilde{u}_a(k) + \begin{bmatrix} u(k-1) \\ 0_{n_u(N_p-1) \times 1} \end{bmatrix} \right) = \Theta \Delta \tilde{u}(k) + \Upsilon u(k-1)$$

It is immediate to find

$$\tilde{\eta}(k) = \Psi x(k) + \Upsilon u(k-1) + \Theta \Delta \tilde{u}(k)$$

Performance indexes

Now, define

$$\tilde{c}_{io}(k) = \tilde{r}(k) - \tilde{\eta}(k) + \Theta_{io}\tilde{u}(k) = \tilde{r}(k) - \Psi x(k)$$

$$\tilde{c}_{iio}(k) = \tilde{r}(k) - \tilde{\eta}(k) + \Theta_{iio}\Delta\tilde{u}(k) = \tilde{r}(k) - \Psi x(k)$$

$$\tilde{c}(k) = \tilde{r}(k) - \tilde{\eta}(k) + \Theta\Delta\tilde{u}(k) = \tilde{r}(k) - \Psi x(k) - \Upsilon u(k-1)$$

Then, it is immediate to find the performance indexes for each case:

$$\tilde{z}^T(k)\tilde{\Omega}\tilde{z}(k) = \tilde{z}_1^T(k)\tilde{\Omega}_1\tilde{z}_1(k) + \tilde{z}_2^T(k)\tilde{\Omega}_2\tilde{z}_2(k)$$

Performance indexes

e.g.

We find

$$\begin{aligned}\tilde{z}^T(k)\tilde{\Omega}\tilde{z}(k) &= \tilde{z}_1^T(k)\tilde{\Omega}_1\tilde{z}_1(k) + \tilde{z}_2^T(k)\tilde{\Omega}_2\tilde{z}_2(k) \\ &= (\tilde{\eta}(k) - \tilde{r}(k))^T Q (\tilde{\eta}(k) - \tilde{r}(k)) + \Delta\tilde{u}^T(k)R\Delta\tilde{u}(k) \\ &= (\Theta\Delta\tilde{u}(k) - \tilde{c}(k))^T Q (\Theta\Delta\tilde{u}(k) - \tilde{c}(k)) + \Delta\tilde{u}^T(k)R\Delta\tilde{u}(k)\end{aligned}$$

s.t.

$$\tilde{z}^T(k)\tilde{\Omega}\tilde{z}(k) = \frac{1}{2}\Delta\tilde{u}^T(k)H\Delta\tilde{u}(k) + \Delta\tilde{u}^T(k)f + f_o$$

where

$$H = 2(\Theta^T Q \Theta + R), \quad f = -2\Theta^T Q \tilde{c}(k), \quad f_o = \tilde{c}^T(k)Q\tilde{c}(k).$$

Constraints

- Inequality constraints

$$u_{\min} \leq u(k) \leq u_{\max}, \quad \Delta u_{\min} \leq \Delta u(k) \leq \Delta u_{\max}$$

$$\eta_{\min} \leq \eta(k) \leq \eta_{\max}, \quad x_{\min} \leq x(k) \leq x_{\max}$$

$$A\tilde{v}(k) \leq b(k)$$

- Equality constraints: motivated by control algorithm itself

Control horizon constraint: $\Delta u(k+i|k) = 0$ for $i \geq N_c$

The state end-point constraint: $\eta(k+N_p-1|k) = \eta_{ss}$

Note:

$$u(k+i) = u(k+N_c-1) \quad \text{for } i \geq N_c-1 \quad \text{s.t. } u(k+N_c-1) = \dots = u(k+N_p-1)$$

Constraints

- Constraints for the output:

$$\underline{\eta} \leq \eta \leq \bar{\eta}.$$

- The constraints on the control and its rate:

$$\underline{u} \leq u(k+i) \leq \bar{u}$$

and

$$\underline{\Delta u} \leq \Delta u(k+i) \leq \overline{\Delta u}$$

for $i = 0, 1, \dots, N_c - 1$.

- Constraint associated with the control horizon N_c :

$$\Delta u(k+i) = 0, \quad i = N_c, \dots, N_p - 1$$

Note: We use inequalities between vectors as the element-wise inequalities (i.e., $\underline{\eta} \leq \eta \leq \bar{\eta} \Leftrightarrow \underline{\eta}_j \leq \eta_j \leq \bar{\eta}_j$ for $j = 1, \dots, n_\eta$).

Constraints

Multiple inequality constraints can be combined by stacking

$$\left. \begin{array}{l} \eta_1(k) \leq \bar{\eta}_1(k) \\ \vdots \\ \eta_m(k) \leq \bar{\eta}_m(k) \end{array} \right\} \Rightarrow \eta(k) = \begin{bmatrix} \eta_1(k) \\ \vdots \\ \eta_m(k) \end{bmatrix} \leq \begin{bmatrix} \bar{\eta}_1(k) \\ \vdots \\ \bar{\eta}_m(k) \end{bmatrix} = \bar{\eta}(k)$$

Constraints

Two-sided inequality constraint

$$\underline{\eta}_1(k) \leq \eta_1(k) \leq \bar{\eta}_1(k)$$

can be translated into one-sided inequality constraint

$$\eta(k) = \begin{bmatrix} \eta_1(k) \\ -\eta_1(k) \end{bmatrix} \leq \begin{bmatrix} \bar{\eta}_1(k) \\ -\underline{\eta}_1(k) \end{bmatrix} = \bar{\eta}(k)$$

Constraints

Assumed that the constraints over outputs, inputs and actuator slew rates are given by

$$G_{\eta} \tilde{\eta}(k) \leq g_1$$

$$G_u \tilde{u}(k) \leq g_2$$

$$G_{\Delta u} \Delta \tilde{u}(k) \leq g_3$$

where G_{η} , G_u , and $G_{\Delta u}$ are matrices to represent each constraint.

Then, the constraints need to be described in terms of the decision variable \tilde{v} (\tilde{u} or $\Delta \tilde{u}$)

Note: For future values we use the predicted values

Constraints

e.g. $G_\eta \tilde{\eta}(k) \leq g_1$ in terms of $\tilde{v}(k)$

Using $\tilde{\eta}(k) = \Psi x(k) + \Theta \Delta \tilde{u}(k)$ leads to

$$G_\eta \tilde{\eta}(k) = G_\eta (\Xi \eta(k) + \Theta \Delta \tilde{u}(k)) \leq g_1$$

Using $\tilde{\eta}(k) = \Psi x(k) + \Theta_{io} \tilde{u}(k)$ leads to

$$G_\eta \tilde{\eta}(k) = G_\eta (\Xi \eta(k) + \Upsilon u(k-1) + \Theta_{io} \tilde{u}(k)) \leq g_1$$

Using $\tilde{\eta}(k) = \Psi x(k) + \Upsilon u(k-1) + \Theta_{iio} \Delta \tilde{u}(k)$ leads to

$$G_\eta \tilde{\eta}(k) = G_\eta (\Xi \eta(k) + \Upsilon u(k-1) + \Theta_{iio} \Delta \tilde{u}(k)) \leq g_1$$

Constraints

e.g. $G_u \tilde{u}(k) \leq g_2$ in terms of $\Delta \tilde{u}(k)$

we find

$$\tilde{u}(k) = \left(I_{n_u N_c} - \begin{bmatrix} 0 & 0 \\ I_{n_u(N_c-1)} & 0 \end{bmatrix} \right)^{-1} \Delta \tilde{u}(k) + \begin{bmatrix} u(k-1) \\ \vdots \\ u(k-1) \end{bmatrix}$$

Thus

$$G_u \left(I_{n_u N_c} - \begin{bmatrix} 0 & 0 \\ I_{n_u(N_c-1)} & 0 \end{bmatrix} \right)^{-1} \Delta \tilde{u}(k) + G_u \begin{bmatrix} u(k-1) \\ \vdots \\ u(k-1) \end{bmatrix} = F_d \Delta \tilde{u}(k) + F_o u(k-1) \leq g_2$$

$$F_d \Delta \tilde{u}(k) \leq g'_2 = g_2 - F_o u(k-1)$$

Constraints

e.g. $G_{\Delta u} \Delta \tilde{u}(k) \leq g_3$ in terms of $\tilde{u}(k)$

we find

$$\Delta \tilde{u}(k) = \left(I_{n_u N_c} - \begin{bmatrix} 0 & 0 \\ I_{n_u(N_c-1)} & 0 \end{bmatrix} \right) \tilde{u}(k) - \begin{bmatrix} u(k-1) \\ 0_{n_u(N_c-1) \times 1} \end{bmatrix}$$

Thus

$$G_{\Delta u} \Delta \tilde{u}(k) = G_{\Delta u} \left(I_{n_u N_c} - \begin{bmatrix} 0 & 0 \\ I_{n_u(N_c-1)} & 0 \end{bmatrix} \right) \tilde{u}(k) - G_{\Delta u} \begin{bmatrix} u(k-1) \\ 0_{n_u(N_c-1) \times 1} \end{bmatrix} \leq g_3$$

$$G_{\Delta u} \left(I_{n_u N_c} - \begin{bmatrix} 0 & 0 \\ I_{n_u(N_c-1)} & 0 \end{bmatrix} \right) \tilde{u}(k) \leq g'_3 = g_3 + G_{\Delta u} \begin{bmatrix} u(k-1) \\ 0_{n_u(N_c-1) \times 1} \end{bmatrix}$$

Constraints

The constraints can then be rewritten as

$$A\Delta\tilde{u}(k) \leq b(k) = b_o + B \begin{bmatrix} \eta(k) \\ u(k-1) \\ r(k) \end{bmatrix}$$

with appropriate matrices and vector A , B , and b_o .

Standard MPC Problem

The standard MPC performance index at time k :

$$\tilde{z}^T(k) \tilde{\Omega} \tilde{z}(k) = \sum_{i=N_m}^{N_p-1} z_1^T(k+i) \Omega_1(i) z_1(k+i) + \sum_{i=0}^{N_c-1} z_2^T(k+i) \Omega_2(i) z_2(k+i)$$

The MPC optimization problem

$$\begin{aligned} \min_{\tilde{v}} \tilde{z}^T(k) \tilde{\Omega} \tilde{z}(k) &= \frac{1}{2} \tilde{v}^T(k) H \tilde{v}^T(k) + \tilde{v}^T(k) f + f_o \\ \text{s.t. } A \tilde{v}(k) &\leq b(k) \end{aligned}$$

Then, the optimal control is

$$u^*(k) = v^*(k) \text{ if } v = u$$

$$u^*(k) = (1 - q^{-1})^{-1} \Delta u^*(k) = \Delta u^*(k) + u(k-1) = v^*(k) + u(k-1) \text{ if } v = \Delta u$$

Standard MPC Problem

Given $\eta(k)$, $r(k)$, $u(k-1)$, find $\tilde{u}^*(k)$ or $\Delta\tilde{u}^*(k)$

through the optimization problem

$$\begin{aligned} \min_{\tilde{v}} \tilde{z}^T(k) \tilde{\Omega} \tilde{z}(k) &= \frac{1}{2} \tilde{v}^T(k) H \tilde{v}^T(k) + \tilde{v}^T(k) f + f_o \\ \text{s.t. } A\tilde{v} &\leq b \end{aligned}$$

Then, the optimal control is

$$u^*(k) = v^*(k) \text{ if } v = u$$

$$u^*(k) = \Delta u^*(k) + u(k-1) = v^*(k) + u(k-1) \text{ if } v = \Delta u$$

Unconstrained MPC problem

Given $\eta(k)$, $r(k)$, $u(k-1)$, find $\tilde{u}^*(k)$ or $\Delta\tilde{u}^*(k)$

through the optimization problem

$$\min_{\tilde{v}} \tilde{z}^T(k) \tilde{\Omega} \tilde{z}(k) = \frac{1}{2} \tilde{v}^T(k) H \tilde{v}^T(k) + \tilde{v}^T(k) f + f_o$$

Then, the optimal soln. is

$$\tilde{v}^*(k) = -H^{-1} f$$

and the optimal control is

$$u^*(k) = v^*(k) \text{ if } v = u$$

$$u^*(k) = \Delta u^*(k) + u(k-1) = v^*(k) + u(k-1) \text{ if } v = \Delta u$$

Constrained MPC problem

Given $\eta(k)$, $r(k)$, $u(k-1)$, find $\tilde{u}^*(k)$ or $\Delta\tilde{u}^*(k)$

through the optimization problem

$$\begin{aligned} \min_{\tilde{v}} \tilde{z}^T(k) \tilde{\Omega} \tilde{z}(k) &= \frac{1}{2} \tilde{v}^T(k) H \tilde{v}^T(k) + \tilde{v}^T(k) f + f_o \\ \text{s.t. } A\tilde{v} &\leq b \end{aligned}$$

The optimal soln. $\tilde{v}^*(k)$ is computed using QP solvers

Then, the optimal control is

$$u^*(k) = v^*(k) \text{ if } v = u$$

$$u^*(k) = \Delta u^*(k) + u(k-1) = v^*(k) + u(k-1) \text{ if } v = \Delta u$$

Stability

Assume

the model is perfect

(Φ, Γ_2) controllable

(Φ, C_1) observable

Can the optimal solution guarantee closed loop stability?

How to attain closed loop stability?

- _ Terminal constraints
- _ Infinite horizon

Terminal constraints ensures stability

Let the terminal constraint $x(k + N_p - 1) = 0$.

Consider

$$V(k) = \tilde{z}^T(k) \tilde{z}(k) = \sum_{i=0}^{N_p-1} J(x(k+i), u(k+i)) = \sum_{i=0}^{N_p-1} z^T(k+i) z(k+i)$$

where

$$J(x(\cdot), u(\cdot)) = z^T(\cdot) z(\cdot) \geq 0$$

and $J(x(\cdot), u(\cdot)) = 0$ only if $x = 0$ and $u = 0$.

Let $V^*(k)$ be the optimal value of $V(k)$ with the optimizer $u^*(k)$.

Clearly, $V^*(k) \geq 0$ and $V^*(k) = 0$ only if $x(k) = 0$ (then, the optimal soln. is to set $u(k+i) = 0$ for all i)

Terminal constraints ensures stability

Now, consider

$$V(k+1) = \tilde{z}^T(k+1)\tilde{z}(k+1) = \sum_{i=0}^{N_p-1} J(x(k+1+i), u(k+i))$$

Note

$$\begin{aligned} V^*(k+1) &= \min_u \tilde{z}^T(k+1)\tilde{z}(k+1) = \min_u \sum_{i=0}^{N_p-1} J(x(k+1+i), u(k+1+i)) \\ &= \min_u \left(\sum_{i=0}^{N_p-1} J(x(k+i), u(k+i)) \right. \\ &\quad \left. + J(x((k+1) + N_p - 1), u((k+1) + N_p - 1)) - J(x(k), u(k)) \right) \\ &\leq V^*(k) \end{aligned}$$

Thus $V^*(k)$ is a Lyapunov fcn. and $(x, u) = (0, 0)$ is stable.

Terminal constraints ensures stability

e.g.

$$x(k+1) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$$
$$z(k) = C_1 x(k) + D_{12} u(k) = \begin{bmatrix} 2 & 0 \end{bmatrix} x(k) + u(k)$$

Let $N_p = 1$.

$$z^T(k) = z(k) = x^T(k) C_1^T C_1 x(k) + u^T(k) D_{12}^T D_{12} u(k) + 2x^T(k) C_1^T D_{12} u(k)$$

$$\frac{\partial z^T z}{\partial u} = D_{12}^T D_{12} u + D_{12}^T C_1 x = 0$$

$$u = -(D_{12}^T D_{12})^{-1} C_1 x = \begin{bmatrix} -2 & 0 \end{bmatrix} x(k)$$

Terminal constraints ensures stability

The closed loop becomes

$$x(k+1) = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} x(k)$$

The optimal soln. results in clp instability

Add a terminal constraint $x(k + N_p - 1) = \dots = 0$. Then

$$x(k+1) = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} x(k) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Adding a terminal constraint can guarantee clp stability

Exercise: What happens if we choose $N_p = 2$

Terminal constraint set

Terminal constraint set: \mathcal{X}_f

Use PC to derive the states into \mathcal{X}_f that includes the origin.

All the constraints become inactive in \mathcal{X}_f

Use some other controller that guarantees stability: “Dual-mode PC”

Note: All MPC which guarantee clp stability have terminal sets.

Infinite horizons: principle of optimality

Finite horizon:

Principle of optimality does not apply because there a different optimization problem arises at each step.

At time k , an optimal trj. is computed over the prediction horizon of length N_p .

At time $k + 1$, with a perfect model, $x(k + 1) = x(k + 1|k)$.

However, a new information $x(k + N_p)$ enters and may lead to an optimal trj. very different from the one computed at time k , which had not been considered at time k

Infinite horizon

Principle of optimality applies: the optimal trj. is not changing

At time k , an optimal trj. over the whole prediction horizon is determined

At time $k + 1$, no new time interval enters the optimization, so the optimal trj. is not changed:

Infinite horizon gives stability:

For a stable system

$$x(k+1) = \Phi x(k) + \Gamma_2 v(k)$$

$$z_1(k) = C_1 x(k)$$

$$z_2(k) = D_{12} v(k)$$

Conditions for control:

$$\Delta u(k+i) = 0 \quad \text{for } i \geq N_p - 1$$

$$u(k+i) = u(k+i-1) \quad \text{for } i \geq N_p - 1$$

$$u(k+i) = 0 \quad \text{for } i \geq N_p - 1 \quad (\text{zero steady state})$$

Infinite horizon gives stability:

Let us consider the IIO case: $v(\cdot) = \Delta u(\cdot)$

$$\begin{aligned} V(k) &= \sum_{i=0}^{\infty} z^T(k+i)z(k+i) \\ &= \sum_{i=0}^{\infty} (\eta^T(k+i)\eta(k+i) + v^T(k+i)R(i)v(k+i)) \\ &= \sum_{i=0}^{\infty} x^T(k+i)Qx(k+i) + \sum_{i=0}^{N_p-1} \Delta u^T(k+i)R\Delta u(k+i) \end{aligned}$$

$$V^*(k) = \sum_{i=0}^{\infty} (\eta^*(k+i))^T \eta^*(k+i) + \sum_{i=0}^{N_p-1} (\Delta u^*(k+i))^T R \Delta u^*(k+i)$$

Infinite horizon gives stability:

$$\begin{aligned} V(k+1) &= \sum_{i=0}^{\infty} \eta^T(k+1+i)\eta(k+1+i) + \sum_{i=0}^{N_p-1} \Delta u^T(k+1+i)R\Delta u(k+1+i) \\ &= \sum_{i=0}^{\infty} \eta^T(k+i)\eta(k+i) - \eta^T(k)\eta(k) \\ &\quad + \sum_{i=0}^{N_p-1} \Delta u^T(k+i)R\Delta u(k+i) + \Delta u^T(k+N_p)R\Delta u(k+N_p) \\ &\quad - \Delta u^T(k)R\Delta u(k) \\ &= \sum_{i=0}^{\infty} \eta^T(k+i)\eta(k+i) - \eta^T(k)\eta(k) \\ &\quad + \sum_{i=0}^{N_p-1} \Delta u^T(k+i)R\Delta u(k+i) - \Delta u^T(k)R\Delta u(k) \\ &= V(k) - \eta^T(k)\eta(k) - \Delta u^T(k)R\Delta u(k) \end{aligned}$$

Infinite horizon gives stability:

Thus

$$V^*(k+1) \leq V^*(k)$$

which implies $\|x(k)\|$ decreasing.

The condition (Φ, C_1) observable (x observable from z_1) implies $\|x^*(k)\|$ is decreasing.

Thus $V^*(k)$ is a Lyapunov function for the closed loop, which shows that the clp is stable.

For a unstable system, the unstable mode must be driven to 0 within N_p steps, which are uncontrolled for $i \geq N_p - 1$ s.t. the cost becomes infinite

the unstable modes must be controllable

$$N_p \geq \# \text{ unstable modes}$$

Constraints and infinite horizons

For a stable system, let us consider the cost fcn:

$$\begin{aligned} V(k) &= \sum_{i=0}^{\infty} \eta^T(k+i)\eta(k+i) + \Delta u^T(k+i)R\Delta u(k+i) \\ &= \sum_{i=N_p}^{\infty} \eta^T(k+i)\eta(k+i) \\ &\quad + \sum_{i=0}^{N_p-1} (\eta^T(k+i)\eta(k+i) + \Delta u^T(k+i)R\Delta u(k+i)) \end{aligned}$$

Constraints and infinite horizons

Since $\Delta u(k+i) = 0$ for $i \geq N_p - 1$, we have

$$z(k + N_p) = C_1 x(k + N_p)$$

$$\vdots$$

$$z(k + N_p + j) = C_1 A^j x(k + N_p)$$

s.t.

$$\begin{aligned} \sum_{i=N_p}^{\infty} \eta^T(k+i) \eta(k+i) &= x^T(k + N_p) \left(\sum_{i=0}^{\infty} (\Phi^T)^i C_1^T C_1 \Phi^i \right) x^T(k + N_p) \\ &= x^T(k + N_p) \bar{Q} x^T(k + N_p) \end{aligned}$$

where

$$\bar{Q} = \sum_{i=0}^{\infty} (\Phi^T)^i C_1^T C_1 \Phi^i$$

Constraints and infinite horizons

$$\begin{aligned} V(k) &= \sum_{i=0}^{\infty} \eta^T(k+i)\eta(k+i) + \Delta u^T(k+i)R\Delta u(k+i) \\ &= \sum_{i=N_p}^{\infty} \eta^T(k+i)\eta(k+i) \\ &\quad + \sum_{i=0}^{N_p-1} (\eta^T(k+i)\eta(k+i) + \Delta u^T(k+i)R\Delta u(k+i)) \\ &= x^T(k+N_p)\bar{Q}x(k+N_p) \\ &\quad + \sum_{i=0}^{N_p-1} (\eta^T(k+i)\eta(k+i) + \Delta u^T(k+i)R\Delta u(k+i)) \\ &= x^T(k+N_p)\bar{Q}x(k+N_p) + \tilde{\eta}^T(k)\tilde{\eta}(k) + \Delta\tilde{u}_a^T(k)\tilde{R}\Delta\tilde{u}_a(k) \end{aligned}$$

This now looks like the prediction control problem with a finite horizon of length N_p .

This can be formulated as a standard QP problem.

Constraints and infinite horizons

Note

$$\Phi^T \bar{Q} \Phi = \Phi^T \sum_{i=0}^{\infty} (\Phi^T)^i C_1^T C_1 \Phi^i \Phi = \sum_{i=1}^{\infty} (\Phi^T)^i C_1^T C_1 \Phi^i$$

and

$$\sum_{i=1}^{\infty} (\Phi^T)^i C_1^T C_1 \Phi^i = \sum_{i=0}^{\infty} (\Phi^T)^i C_1^T C_1 \Phi^i - C_1^T C_1 = \bar{Q} - C_1^T C_1$$

Thus

$$\Phi^T \bar{Q} \Phi = \bar{Q} - C_1^T C_1$$

Lyapunov equation: If Φ Shur stable, then $\bar{Q} \geq 0$.

Finite horizon without explicit terminal constraints

Use finite horizon LQ optimal control

When we apply the receding horizon control strategy, we always apply $v^*(k)$ from $\tilde{v}^*(k)$.

We apply the state feedback control law

$$v(k) = -K_{N_p-1}x(k)$$

When can this law be guaranteed to be stabilizing?

When will all the eigenvalues of $\Phi - \Gamma K_{N_p-1}$ be guaranteed to lie inside the unit circle?

Finite horizon without explicit terminal constraints

Reconsider the infinite horizon performance index

$$V(k) = x^T(k + N_p)\bar{Q}x(k + N_p) + \sum_{i=0}^{N_p-1} (\eta^T(k + i)\eta(k + i) + \Delta u^T(k + i)R\Delta u(k + i))$$

Observe the cost

$$V_{N_p}(k) = x^T(k + N_p)P_0x(k + N_p) + \sum_{i=0}^{N_p-1} (x^T(k + i)Qx(k + i) + \Delta u^T(k + i)R\Delta u(k + i))$$

Finding the optimal control sequence which will minimize the finite horizon cost function $V_{N_p}(k)$ can be found from the finite horizon LQ optimal control:

Finite horizon without explicit terminal constraints

The optimal soln. is found as follows:

$$P_{i+1} = \Phi^T P_i \Phi - \Phi^T P_i \Gamma (\Gamma^T P_i \Gamma + R)^{-1} \Gamma^T P_i \Phi + Q$$

$$K_i = (\Gamma^T P_i \Gamma + R)^{-1} \Gamma^T P_i \Phi$$

The optimal control sequence

$$u(k) = -K_{N_p-1} x(k)$$

$$u(k+1) = -K_{N_p-2} x(k+1)$$

$$u(k+i) = -K_{N_p-i-1} x(k+i)$$

produces $V^*(k) = x^T(k) P_{N_p} x(k)$

Finite horizon without explicit terminal constraints

If $N_p \rightarrow \infty$, infinite horizon

$$P_\infty = \Phi^T P_\infty \Phi - \Phi^T P_\infty \Gamma (\Gamma^T P_\infty \Gamma + R)^{-1} \Gamma^T P_\infty \Phi + Q$$

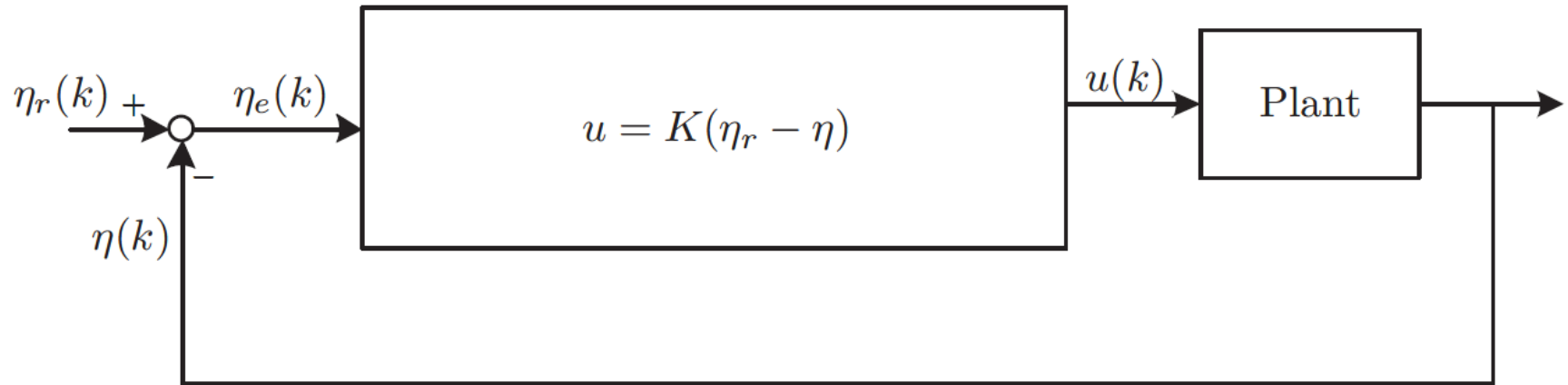
$$K_\infty = (\Gamma^T P_\infty \Gamma + R)^{-1} \Gamma^T P_\infty \Phi$$

$$u(k+i) = -K_\infty x(k+i)$$

This feedback control law is stabilizing (Otherwise, $V_\infty \rightarrow \infty$).

The optimal cost is $V^*(k) = x^T(k) P_\infty x(k)$

Simulation and Implementation



η_r : set point trajectory

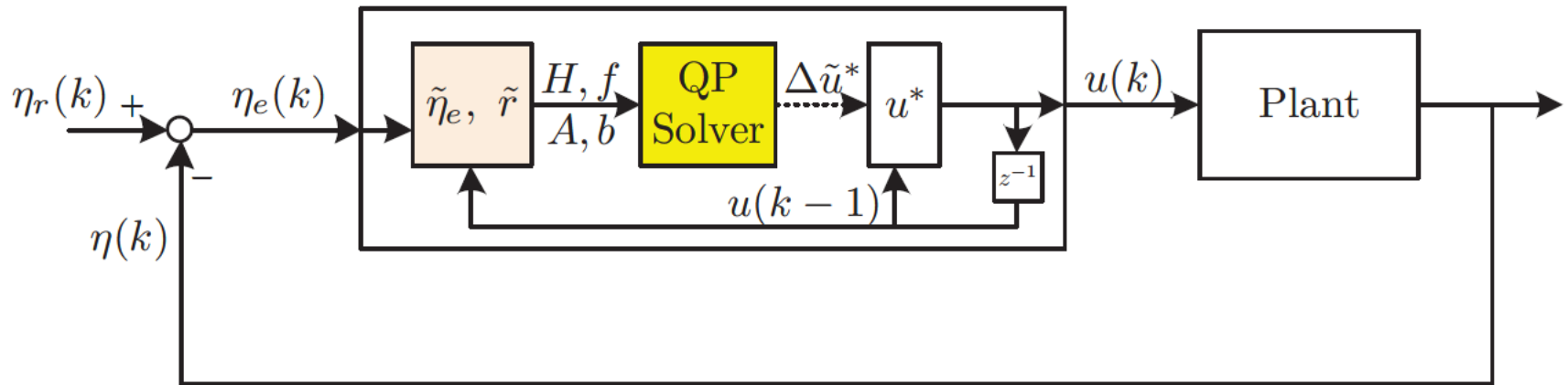
$$\eta_e = \eta_r - \eta$$

$u(k) = K(z)\eta_e(k)$: feedback control

η_e : output & controlled variable of generalized system

Note: Assumed $\eta_e = -\eta + \eta_r = y = C_2x + D_{21}w$, $\eta_r = D_{21}w$, $\eta = -C_2x$.

Simulation and Implementation



η_r : set point trajectory

$$\eta_e = \eta_r - \eta$$

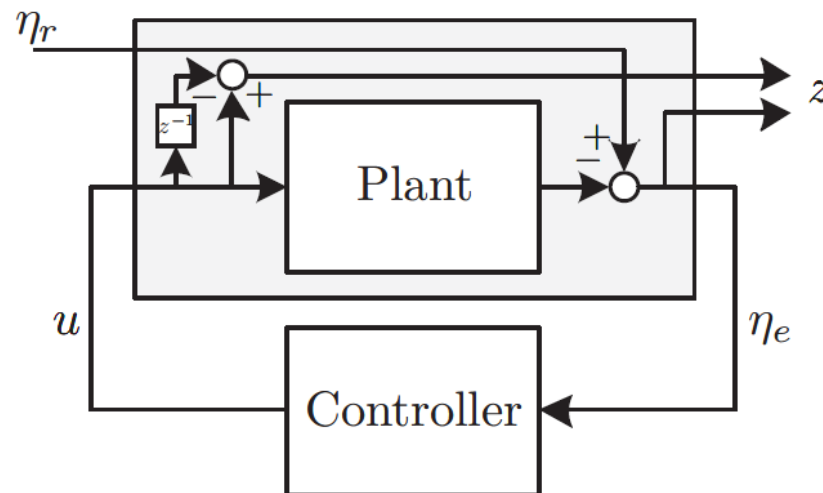
$$u(k) = \boxed{\text{PC}} \eta_e(k) : \text{Predictive controller}$$

η_e : output & controlled variable of generalized system

Note: Assumed $\eta_e = -\eta + \eta_r = y = C_2x + D_{21}w$, $\eta_r = D_{21}w$, $\eta = -C_2x$.

Simulation and Implementation

In the standard control system framework:



Assumed $\eta = y_p$

$\eta_e = y$ available

controlled variables z

exogenous signals w

Simulation and Implementation

The vehicle lateral dynamics in terms of the state vector $x = \begin{bmatrix} e_{yL} & \dot{e}_y & e_\psi & \dot{\psi} \end{bmatrix}^T$:

$$\dot{x} = Ax + B_u u + B_q q = \begin{bmatrix} 0 & 1 & 0 & L \\ 0 & a_{22} & a_{23} & a'_{24} \\ 0 & 0 & 0 & 1 \\ 0 & a'_{42} & a_{43} & a_{44} \end{bmatrix} x + \begin{bmatrix} 0 \\ b'_{21} \\ 0 \\ b_{41} \end{bmatrix} \delta + \begin{bmatrix} -L & V_x \\ -V_x & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix} q$$

where $q = \begin{bmatrix} \dot{\psi}_{des} & e_{\psi L} - e_\psi \end{bmatrix}^T$.

Simulation and Implementation

Discrete-time model:

$$x(k+1) = \Phi x(k) + \Gamma_u u(k) + \Gamma_q q(k)$$

$$z_1(k) = C_1 x(k) - r(k)$$

$$z_2(k) = D_{12} \Delta u(k)$$

$$y(k) = C_2 x(k) + D_{21} w$$

$$\eta_e(k) = C_1 x(k)$$

$$\Delta u(k) = u(k) - u(k-1)$$

$$\text{Assumed } \eta_e = -\eta + \eta_r = y = C_2 x + D_{21} w, \quad \eta_r = D_{21} w, \quad \eta = -C_2 x.$$

Simulation and Implementation

Prediction of the controlled variable:

Assume $\eta_e(k) = \eta_r(k) - \eta(k)$ is available.

Then we use

$$\tilde{\eta}_e(k) = \Psi_{\eta} \eta_e(k) + \Upsilon u(k-1) + \Theta \Delta \tilde{u}(k)$$

Reference trajectory:

We use a simple ref. trj.

$$r(k+i) = e^{-\alpha i} \eta(k), \quad i = 0, \dots, N_p - 1$$

$$\tilde{r}(k) = \Xi \eta(k) = \begin{bmatrix} \vdots \\ e^{-\alpha i} \\ \vdots \end{bmatrix} \eta(k)$$

Simulation and Implementation

Then

$$\tilde{c}(k) = \tilde{r}(k) - \Psi_{\eta} \eta_e(k) - \Upsilon u(k-1)$$

and thus

$$\tilde{\eta}_e(k) - \tilde{r}(k) = \Theta \Delta \tilde{u}(k) - \tilde{c}(k)$$

We then immediately find

$$\tilde{z}^T(k) \tilde{\Omega} \tilde{z}(k) = \frac{1}{2} \Delta \tilde{u}^T(k) H \Delta \tilde{u}(k) + \Delta \tilde{u}^T(k) f + f_o$$

where

$$H = 2(\Theta^T Q \Theta + R), \quad f = -2\Theta^T Q \tilde{c}(k), \quad f_o = \tilde{c}^T(k) Q \tilde{c}(k).$$

Simulation and Implementation

Constraints: controlled variables, steering angle, steering angle rate

- $-\bar{\eta}_e \leq \eta_e(k+i) \leq \bar{\eta}_e$ with $\bar{\eta}_e = \begin{bmatrix} 4 \\ 0.3 \end{bmatrix}$
- $-\bar{u} \leq u(k+i) \leq \bar{u}$ with $\bar{u} = 0.5386$
- $-\overline{\Delta u} \leq \Delta u(k+i) \leq \overline{\Delta u}$ with $\overline{\Delta u} = 0.4987$

We can build a constraint

$$A\Delta\tilde{u}(k) \leq b$$

Now, we are ready to solve the QP problem with the QP performance index and the constraint to find the optimal control solution $\Delta\tilde{u}^*$.