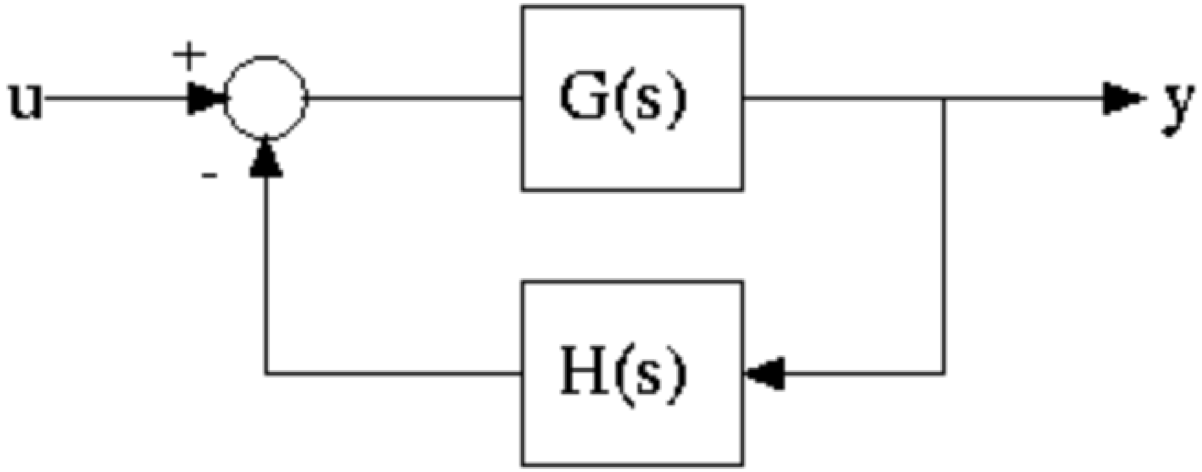
System Control

modeling in the time domain



Modeling in the time domain

- Find state-space representation
- Model electrical & mechanical system in state-space
- Convert a TF to SS
- Convert a SS to TF
- Linearization

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

The general state-space representation

- Linear combination

$$S_1 = K_1x_1 + K_2x_2 + K_3x_3 + \cdots + K_nx_n$$

$$S_2 = K_1x_1x_2 + K_2x_2x_3 + K_3x_3x_4 + \cdots + K_{n-1}x_{n-1}x_n$$

- Linear independence

x_1, x_2, x_3 $\quad$ $x_1, x_2, x_3 = x_1x_2$ $\quad$ $x_1, x_2, x_3 = x_1 + 2x_2$ $\quad$ $x_1, x_2, x_3 = 3x_1 + x_2 + 1$

The general state-space representation

- System variable

  Any variable that responds to an input or initial conditions in a system

- State variables

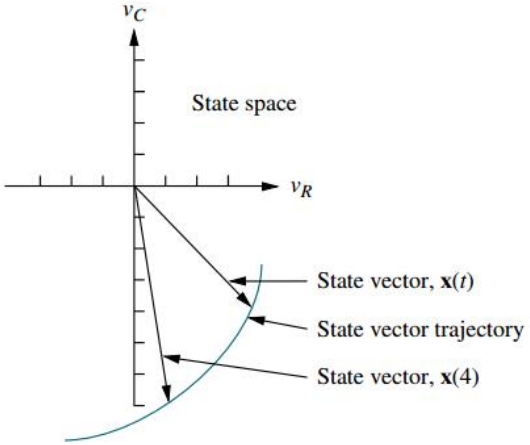
  The **smallest** set of **linearly independent** system variables such that the values of the members of the set at time t_0 along with known forcing functions completely determine the value of all system variables for all $t > t_0$.

- State vector

  A vector whose elements are the state variables

- State space

  The n-dimensional space whose axes are the state variables.



The general state-space representation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$ ← state equation

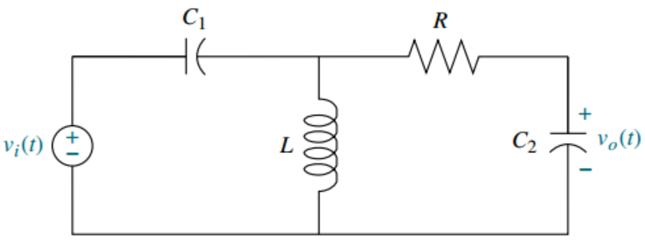
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$ ← output equation

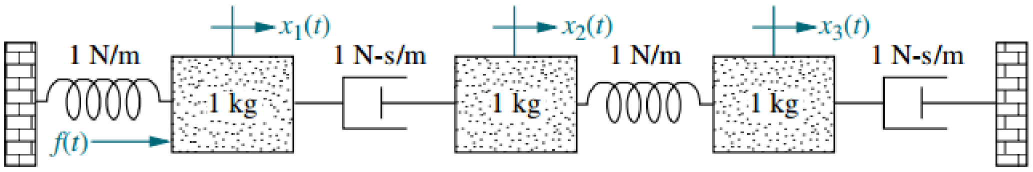
$\mathbf{x}$ = state vector
$\dot{\mathbf{x}}$ = derivative of the state vector with respect to time
$\mathbf{y}$ = output vector
$\mathbf{u}$ = input or control vector
$\mathbf{A}$ = system matrix
$\mathbf{B}$ = input matrix
$\mathbf{C}$ = output matrix
$\mathbf{D}$ = feedforward matrix

State vector, state space

- Minimum number of state variables

Typically, the minimum number required equals the order of the differential equation describing the system.





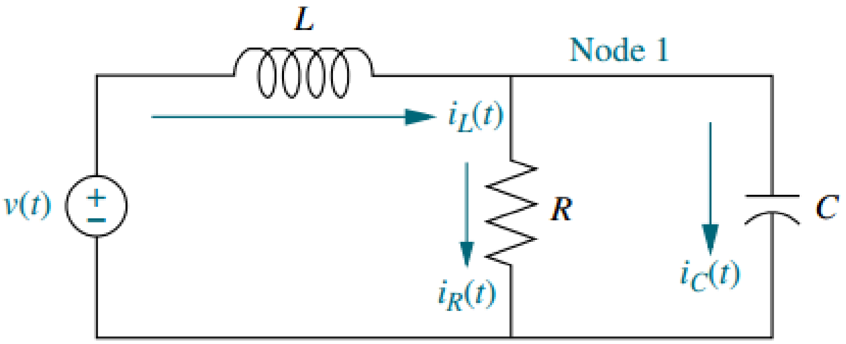
- Linear independence

$$S = K_n x_n + K_{n-1} x_{n-1} + \cdots + K_1 x_1$$

if their linear combination, S, equals zero *only* if every $K_i = 0$ and *no* $x_i = 0$ for all $t \geq 0$.

State-space representation

1. **Label**
2. **Select** the state variables: for all of the energy-storage elements (remember $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$)
3. **Obtain** state equations: represent Eqs in terms of the state variable
4. **Find** output equation (remember $\mathbf{y} = C\mathbf{x} + D\mathbf{u}$)



$$i_R = \frac{1}{R} v_C$$

$$C\frac{dv_C}{dt} = i_C$$

$$L\frac{di_L}{dt} = v_L$$

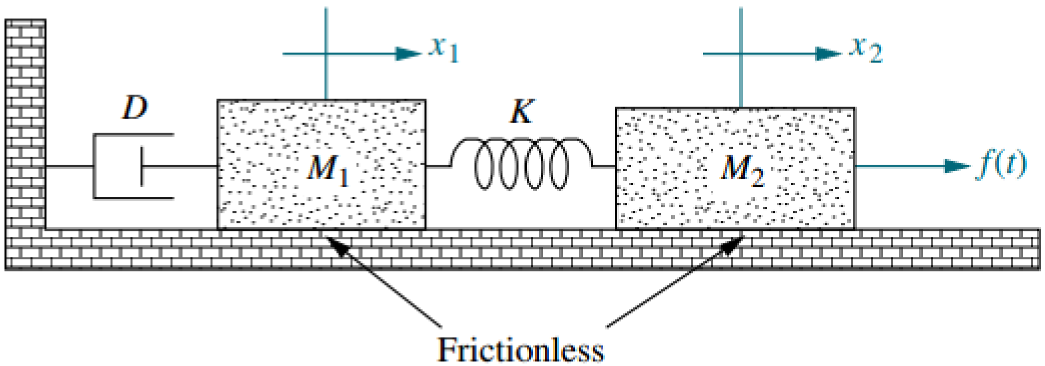
$$\begin{aligned} i_C &= -i_R + i_L \\ &= -\frac{1}{R} v_C + i_L \end{aligned}$$

$$v_L = -v_C + v(t)$$

$$\frac{dv_C}{dt} = -\frac{1}{RC} v_C + \frac{1}{C} i_L$$

$$\frac{di_L}{dt} = -\frac{1}{L} v_C + \frac{1}{L} v(t)$$

Review: Translational mechanical system



M_1 | M_1

$$M_1\frac{d^2x_1}{dt^2}+D\frac{dx_1}{dt}+Kx_1-Kx_2=0$$

M_2 | M_2

$$-Kx_1+M_2\frac{d^2x_2}{dt^2}+Kx_2=f(t)$$

SS

$$\frac{dx_1}{dt}=+v_1$$

$$\frac{dv_1}{dt}=-\frac{K}{M_1}x_1-\frac{D}{M_1}v_1+\frac{K}{M_1}x_2$$

$$\frac{dx_2}{dt}=+v_2$$

$$\frac{dv_2}{dt}=+\frac{K}{M_2}x_1-\frac{K}{M_2}x_2+\frac{1}{M_2}f(t)$$

Converting a TF to SS (1)

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = b_0 u$$

$$x_1 = y, \quad x_2 = \frac{dy}{dt}, \quad x_3 = \frac{d^2 y}{dt^2}, \quad \vdots \quad x_n = \frac{d^{n-1} y}{dt^{n-1}}$$

D

$$\dot{x}_1 = \frac{dy}{dt}, \quad \dot{x}_2 = \frac{d^2 y}{dt^2}, \quad \dot{x}_3 = \frac{d^3 y}{dt^3}, \quad \vdots \quad \dot{x}_n = \frac{d^n y}{dt^n}$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -a_0 x_1 - a_1 x_2 \cdots - a_{n-1} x_n + b_0 u \end{aligned}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} u$$

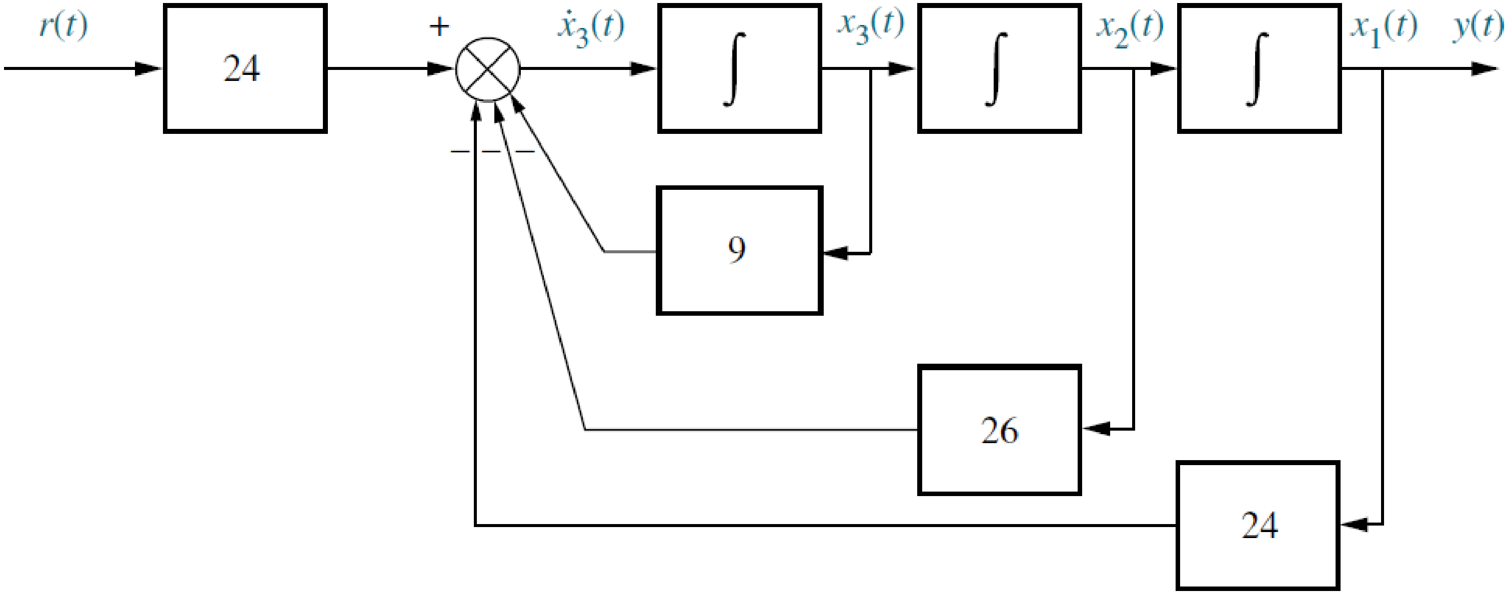
$$y = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Converting a TF to SS (2)

Example

$$\frac{C(s)}{R(s)} = \frac{24}{(s^3 + 9s^2 + 26s + 24)} \quad \Longleftrightarrow \quad \dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r$$

$$x_1 = c$$
$$x_2 = \dot{c}$$
$$x_3 = \ddot{c}$$



Converting a TF to SS (3)

Example

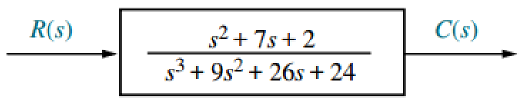
$$\frac{C(s)}{R(s)} = \frac{24}{(s^3 + 9s^2 + 26s + 24)} \quad \Longleftrightarrow \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -24x_1 - 26x_2 - 9x_3 + 24r \\ y &= c = x_1 \end{aligned}$$

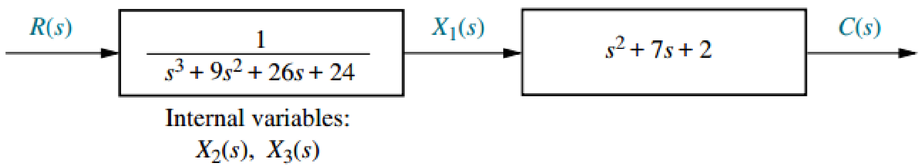
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

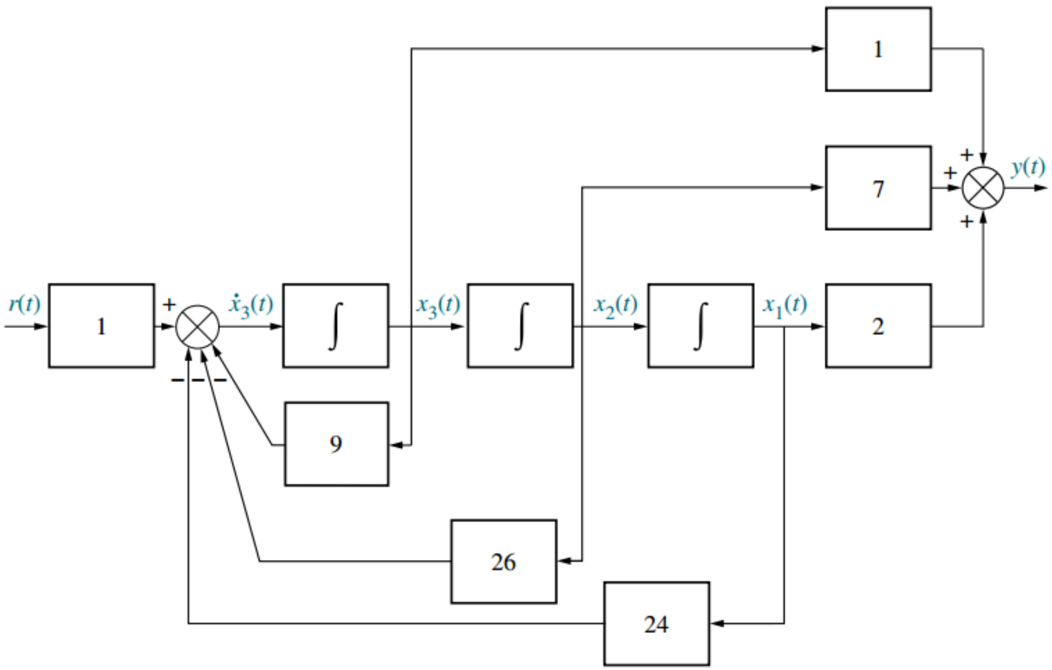
Converting a TF to SS (4)

1. Separate the system into two cascaded blocks





2. Find the state equations



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Converting a SS to TF

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

$\Rightarrow$ $T(s) = Y(s)/U(s)$

Laplace transform

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$$

Solving for $\mathbf{X}(s)$

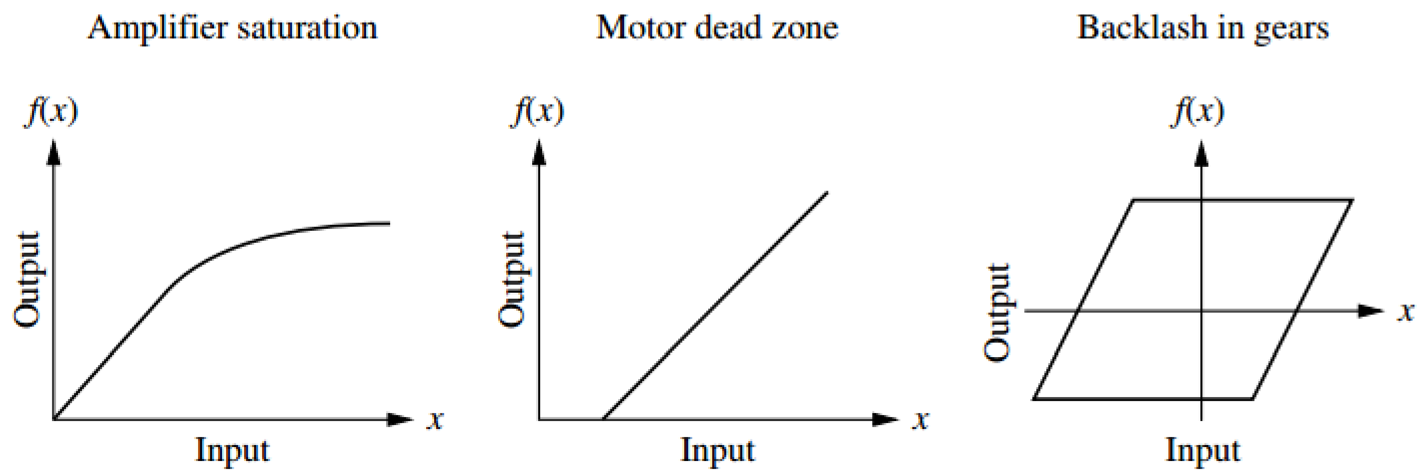
$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

$$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{U}(s)$$

$$T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

Linearization (1)

- Nonlinearities



Linearization (2)

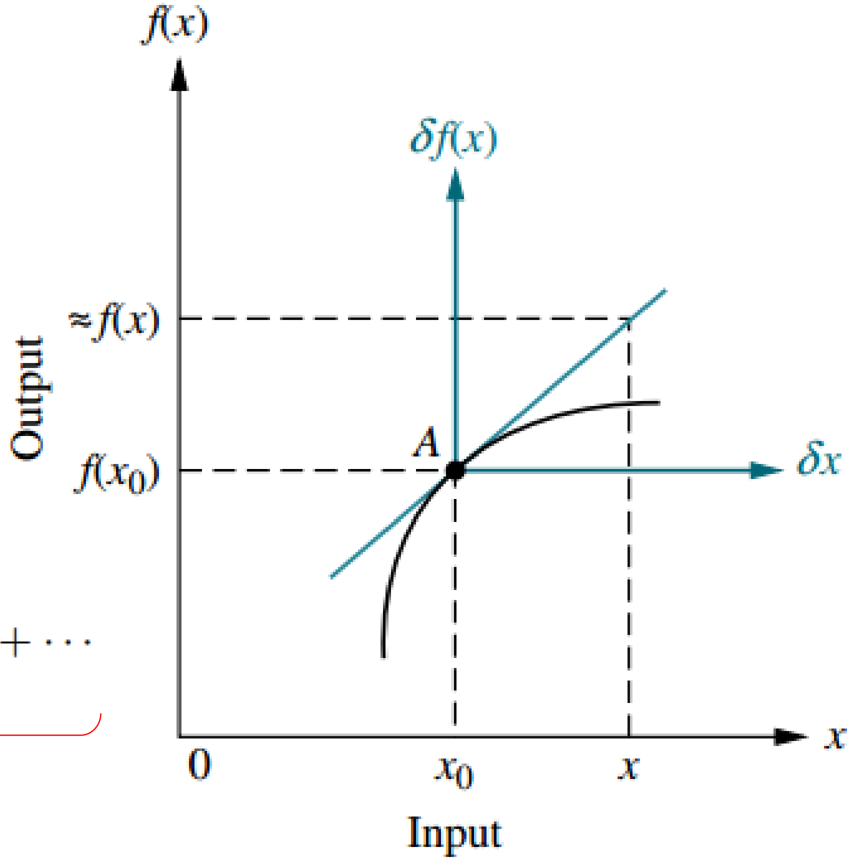
$$[f(x) - f(x_0)] \approx m_a(x - x_0)$$

from which

$$\delta f(x) \approx m_a \delta x$$

and

$$f(x) \approx f(x_0) + m_a(x - x_0) \approx f(x_0) + m_a \delta x$$



Taylor series

$$f(x) = f(x_0) + \left.\frac{df}{dx}\right|_{x=x_0} \frac{(x - x_0)}{1!} + \underbrace{\left.\frac{d^2 f}{dx^2}\right|_{x=x_0} \frac{(x - x_0)^2}{2!} + \cdots}_{h.o.t}$$

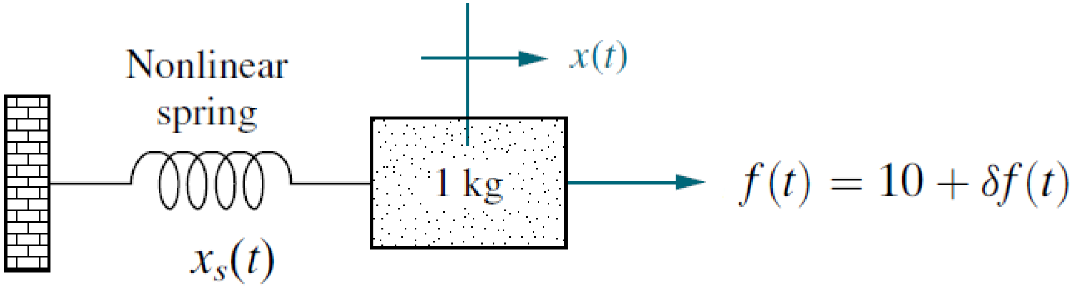
$$f(x) - f(x_0) \approx \left.\frac{df}{dx}\right|_{x=x_0} (x - x_0)$$

or

$$\delta f(x) \approx m|_{x=x_0} \delta x$$

Linearization (3)

- How to get transfer function in nonlinear system?



$f_s(t) = 2x_s^2(t)$ The force of the spring at equilibrium is 10 N.

Summary

- State-space representation
- State variable, vector, space
- State-space ←→ Transfer function
- System linearization