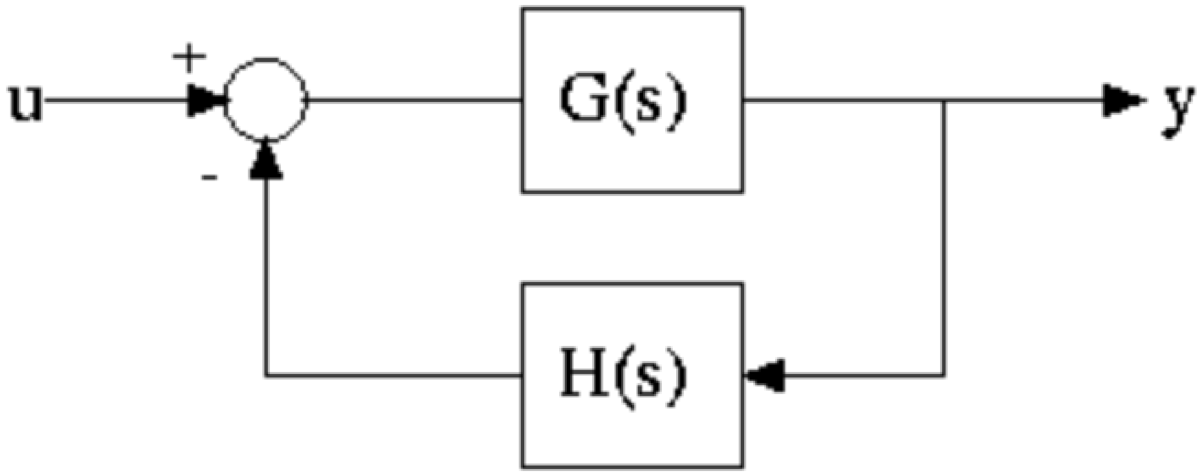
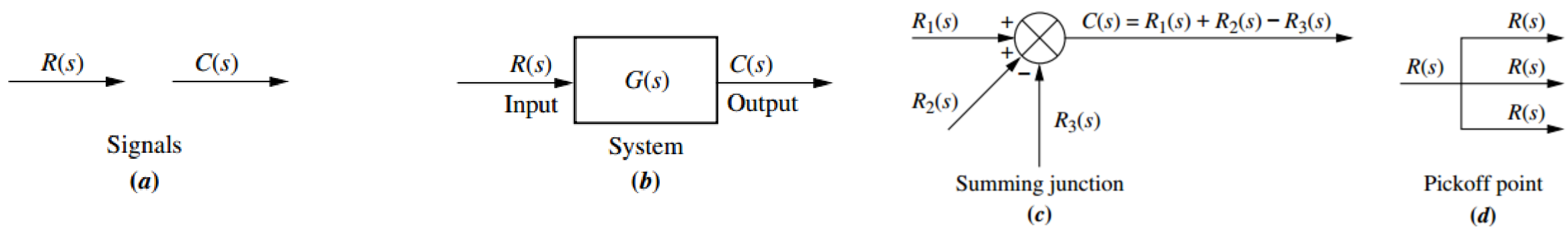
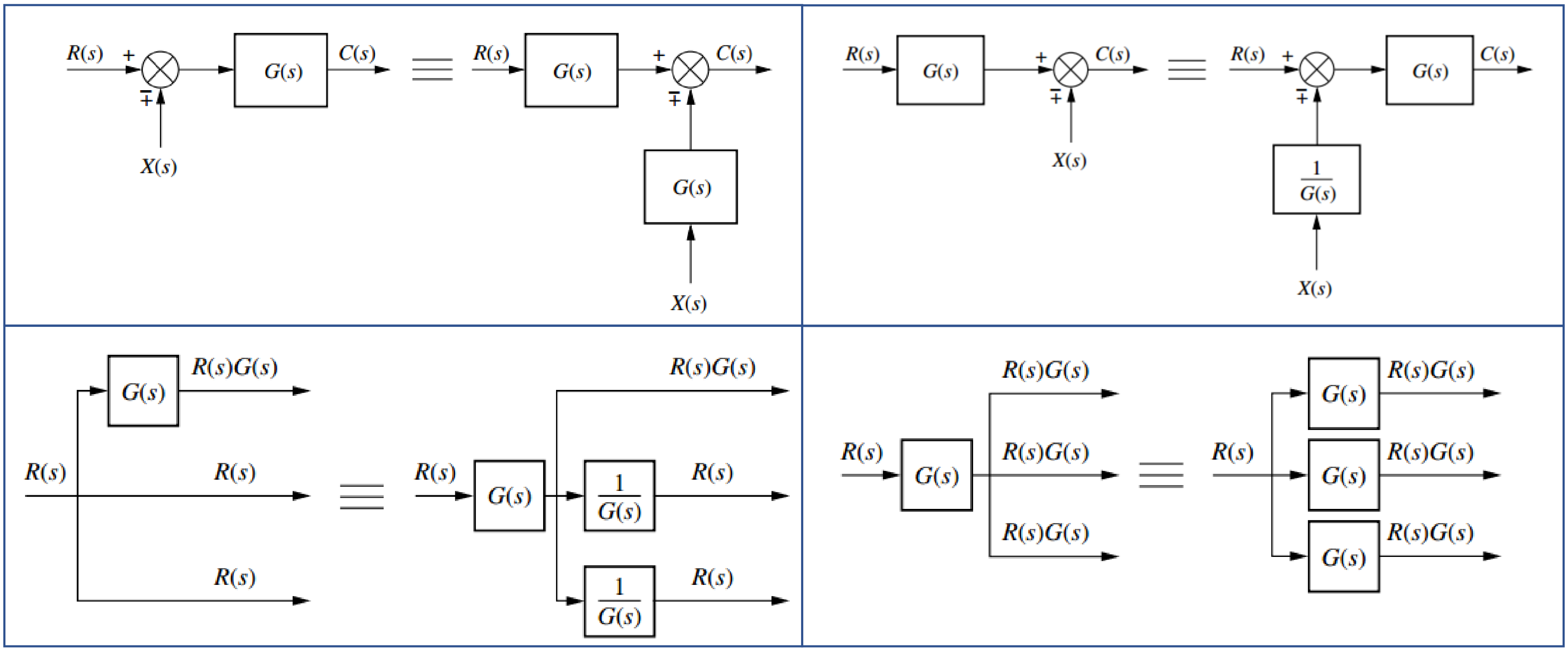
System Control

Reduction of multiple subsystems

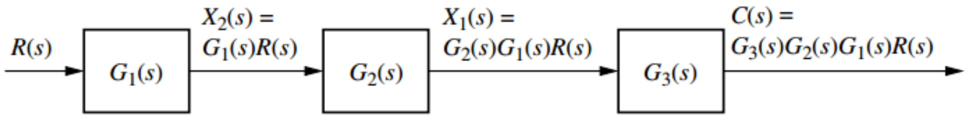


Block diagrams

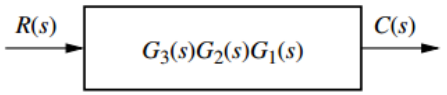




Block diagrams: cascade & parallel

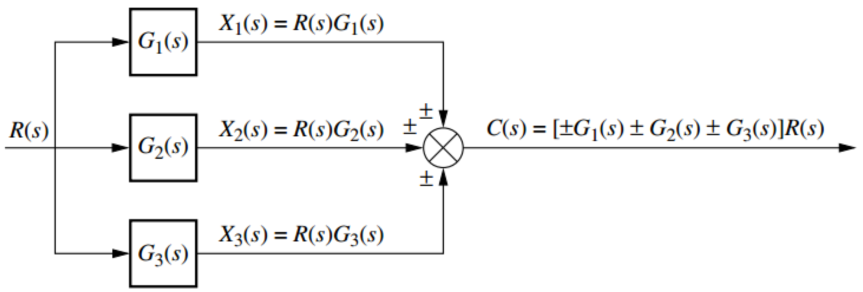


(a)

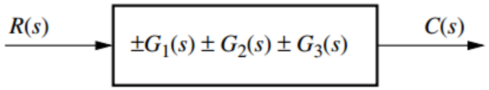


(b)

$$G_e(s) = G_3(s)G_2(s)G_1(s)$$



(a)



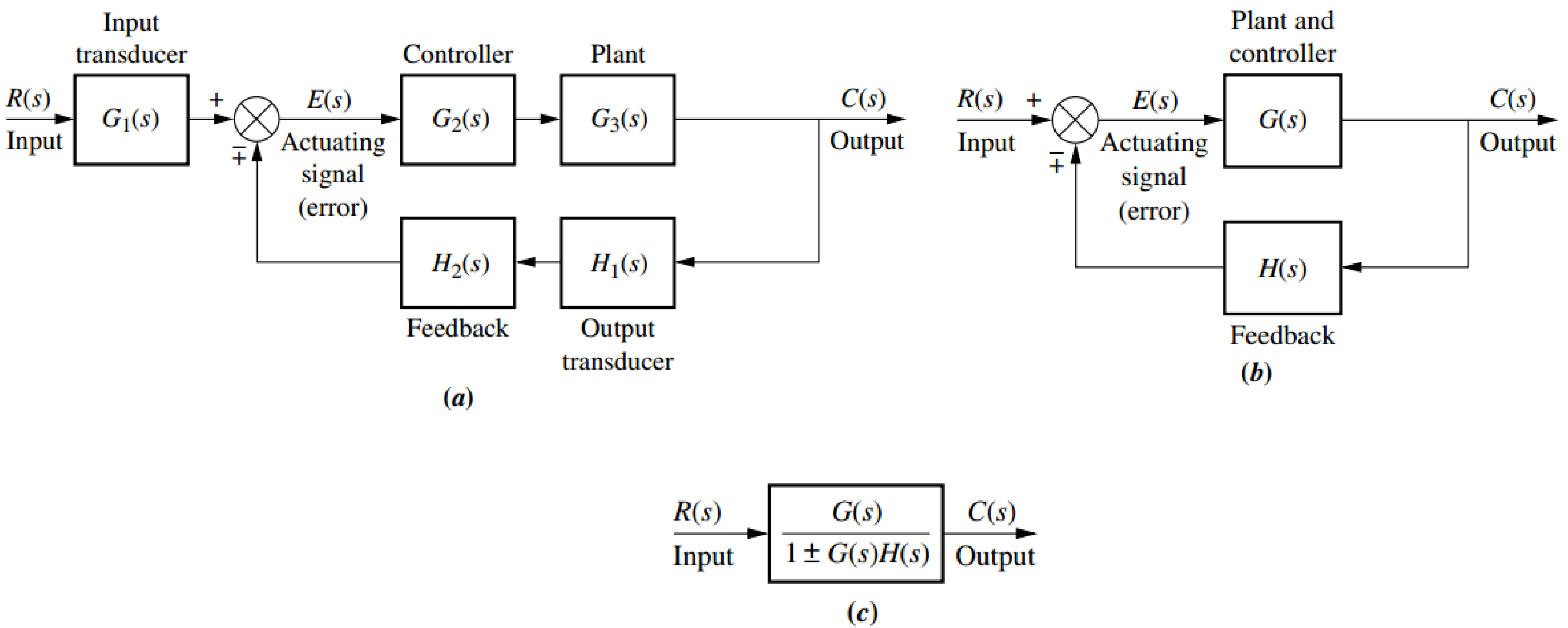
(b)

$$G_e(s) = \pm G_1(s) \pm G_2(s) \pm G_3(s)$$

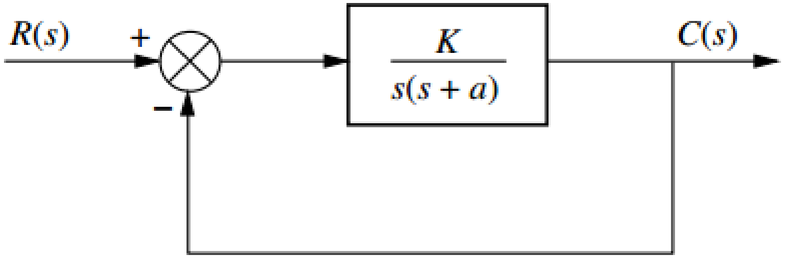
Block diagrams: feedback form

$$E(s) = R(s) \mp C(s)H(s)$$

$$C(s) = E(s)G(s)$$

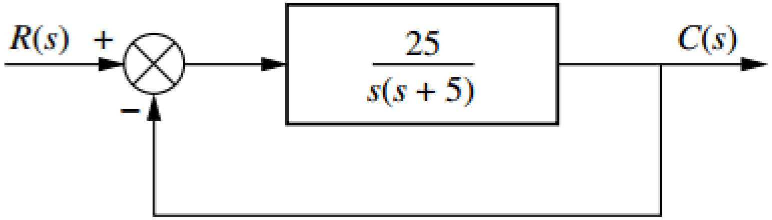


Analysis & design of feedback systems



$$T(s) = \frac{K}{s^2 + as + K} \qquad s_{1,2} = -\frac{a}{2} \pm j\frac{\sqrt{4K - a^2}}{2}$$

Finding Transient Response



$$\omega_n = \sqrt{25} = 5$$

$$2\zeta\omega_n = 5$$

$$\zeta = 0.5$$

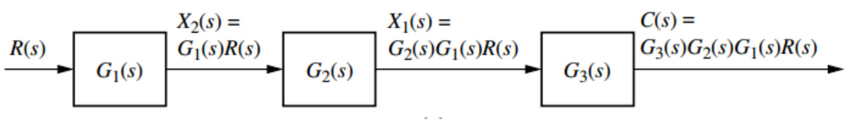
$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.726 \text{ second}$$

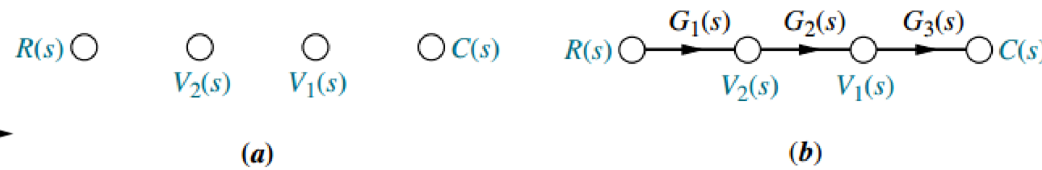
$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100 = 16.303$$

$$T_s = \frac{4}{\zeta\omega_n} = 1.6 \text{ seconds}$$

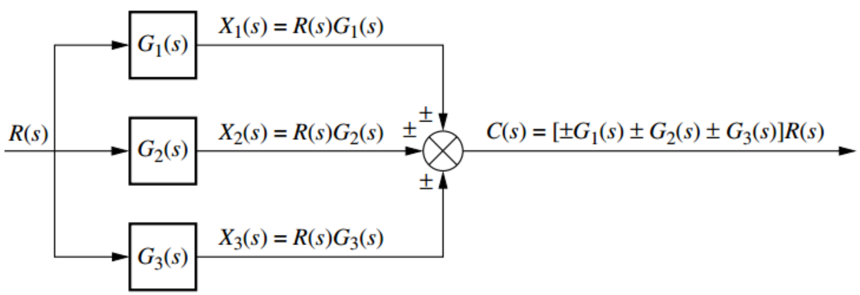
Signal-flow graphs

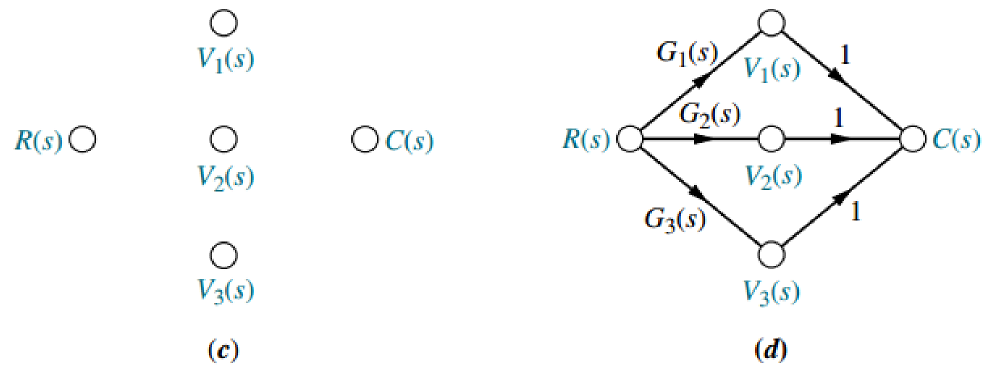
- Cascade



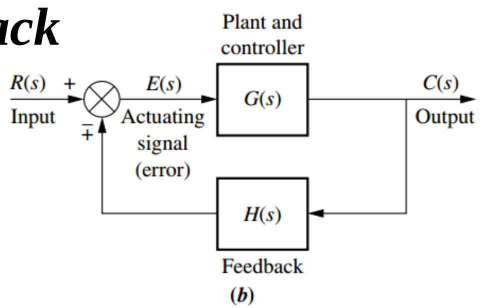


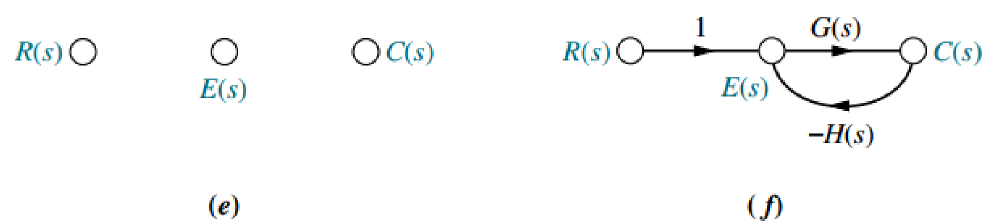
- Parallel





- Feedback





Mason's rule

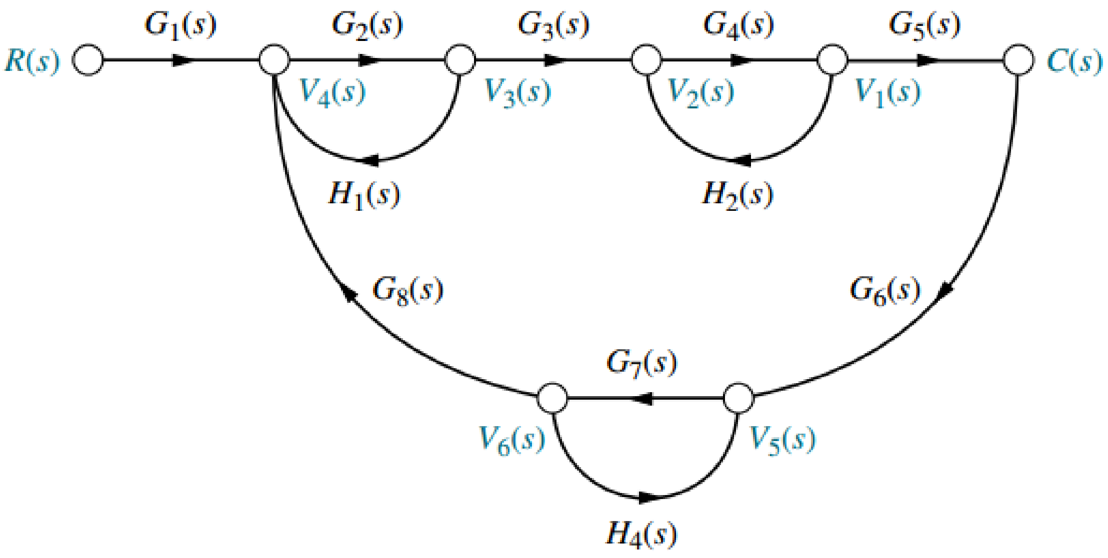
k = number of forward paths

T_k = the kth forward-path gain

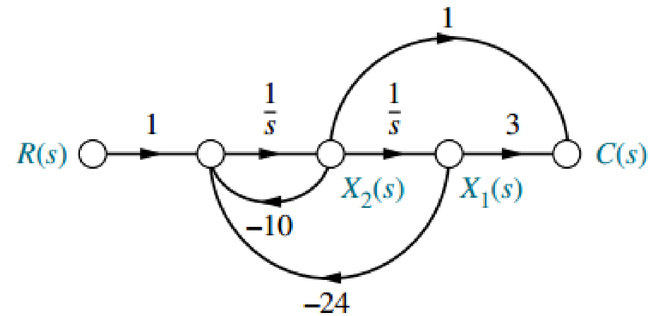
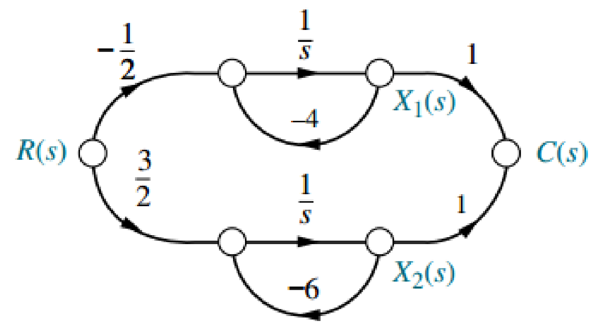
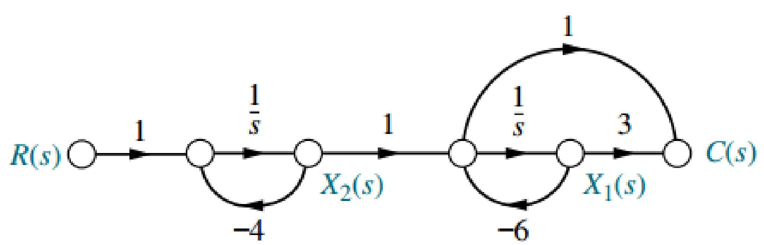
$\Delta = 1 - \Sigma$ loop gains $+ \Sigma$ nontouching-loop gains taken two at a time $- \Sigma$ nontouching-loop gains taken three at a time $+ \Sigma$ nontouching-loop gains taken four at a time $- \ldots$

$\Delta_k = \Delta - \Sigma$ loop gain terms in Δ that touch the kth forward path. In other words, Δ_k is formed by eliminating from Δ those loop gains that touch the kth forward path.

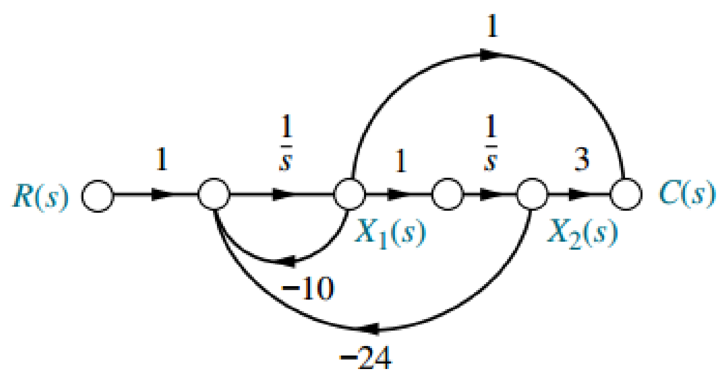
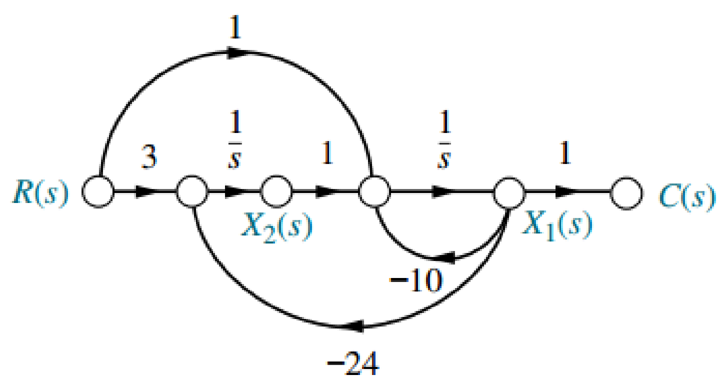
$$G(s) = \frac{C(s)}{R(s)} = \frac{\sum_k T_k \Delta_k}{\Delta}$$



Signal-flow graphs of state equations (1)

Form	Transfer function	Signal-flow diagram	State equations
Phase variable	$\frac{1}{(s^2+10s+24)} * (s+3)$		$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -24 & -10 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$ $y = [3 \;\; 1]\,\mathbf{x}$
Parallel	$\frac{-1/2}{(s+4)} + \frac{3/2}{s+6}$		$\dot{\mathbf{x}} = \begin{bmatrix} -4 & 0 \\ 0 & -6 \end{bmatrix}\mathbf{x} + \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix} r$ $y = [1 \;\; 1]\,\mathbf{x}$
Cascade	$\frac{1}{(s+4)} * \frac{(s+3)}{(s+6)}$		$\dot{\mathbf{x}} = \begin{bmatrix} -6 & 1 \\ 0 & -4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$ $y = [-3 \;\; 1]\,\mathbf{x}$

Signal-flow graphs of state equations (2)

<table>
<tr><th>Form</th><th>Transfer function</th><th>Signal-flow diagram</th><th>State equations</th></tr>
<tr><td>Controller canonical</td><td>$\frac{1}{(s^2 + 10s + 24)} * (s + 3)$</td><td>
</td><td>$\dot{\mathbf{x}} = \begin{bmatrix} -10 & -24 \\ 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r$
$y = [1 \;\; 3]\,\mathbf{x}$</td></tr>
<tr><td>Observer canonical</td><td>$\dfrac{\frac{1}{s} + \frac{3}{s^2}}{1 + \frac{10}{s} + \frac{24}{s^2}}$</td><td>
</td><td>$\dot{\mathbf{x}} = \begin{bmatrix} -10 & 1 \\ -24 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} r$
$y = [1 \;\; 0]\,\mathbf{x}$</td></tr>
</table>

Similarity transformations

$$\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

$$\mathbf{z} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{z} + \mathbf{P}^{-1}\mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{P}\mathbf{z} + \mathbf{D}\mathbf{u}$$

where,

$$\mathbf{P} = [\mathbf{U}_{\mathbf{z}_1}\mathbf{U}_{\mathbf{z}_2}] = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \mathbf{P}\mathbf{z}$$

$$\mathbf{z} = \mathbf{P}^{-1}\mathbf{x}$$

Similarity transformations

Example

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -7 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}$$

$$\begin{aligned} z_1 &= 2x_1 \\ z_2 &= 3x_1 + 2x_2 \\ z_3 &= x_1 + 4x_2 + 5x_3 \end{aligned}$$

$$\mathbf{z} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix} \mathbf{x} = \mathbf{P}^{-1}\mathbf{x}$$

$$\mathbf{P}^{-1}\mathbf{AP} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -7 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0 \\ -0.75 & 0.5 & 0 \\ 0.5 & -0.4 & 0.2 \end{bmatrix} = \begin{bmatrix} -1.5 & 1 & 0 \\ -1.25 & 0.7 & 0.4 \\ -2.5 & 0.4 & -6.2 \end{bmatrix}$$

$$\mathbf{P}^{-1}\mathbf{B} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$$

$$\mathbf{CP} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0 \\ -0.75 & 0.5 & 0 \\ 0.5 & -0.4 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 0 \end{bmatrix}$$

$$\dot{\mathbf{z}} = \begin{bmatrix} -1.5 & 1 & 0 \\ -1.25 & 0.7 & 0.4 \\ -2.55 & 0.4 & -6.2 \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0.5 & 0 & 0 \end{bmatrix} \mathbf{z}$$

Similarity transformations

Example: Diagonalization

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}u$$

$$y = \begin{bmatrix} 2 & 3 \end{bmatrix}\mathbf{x}$$

$$\begin{aligned} \det(\lambda \mathbf{I} - \mathbf{A}) &= \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \right| \\ &= \begin{vmatrix} \lambda + 3 & -1 \\ -1 & \lambda + 3 \end{vmatrix} \\ &= \lambda^2 + 6\lambda + 8 \end{aligned}$$

$\lambda = -2$, and -4.

$$\mathbf{A}\mathbf{x_i} = \lambda \mathbf{x_i}$$

$$\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} c \\ c \end{bmatrix}$$

$$\mathbf{A}\mathbf{x_i} = \lambda \mathbf{x_i}$$

$$\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -4\begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} c \\ -c \end{bmatrix}$$

$$\mathbf{x_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \mathbf{x_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Similarity transformations

Example: Diagonalization

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}u$$

$$y = \begin{bmatrix} 2 & 3 \end{bmatrix}\mathbf{x}$$

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \right| = \begin{vmatrix} \lambda + 3 & -1 \\ -1 & \lambda + 3 \end{vmatrix} = \lambda^2 + 6\lambda + 8$$

$\lambda = -2$, and -4.

$$\mathbf{A}\mathbf{x_i} = \lambda\mathbf{x_i}$$

$$\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} c \\ c \end{bmatrix}$$

$$\mathbf{A}\mathbf{x_i} = \lambda\mathbf{x_i}$$

$$\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -4\begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} c \\ -c \end{bmatrix}$$

$$\mathbf{x_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \mathbf{x_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

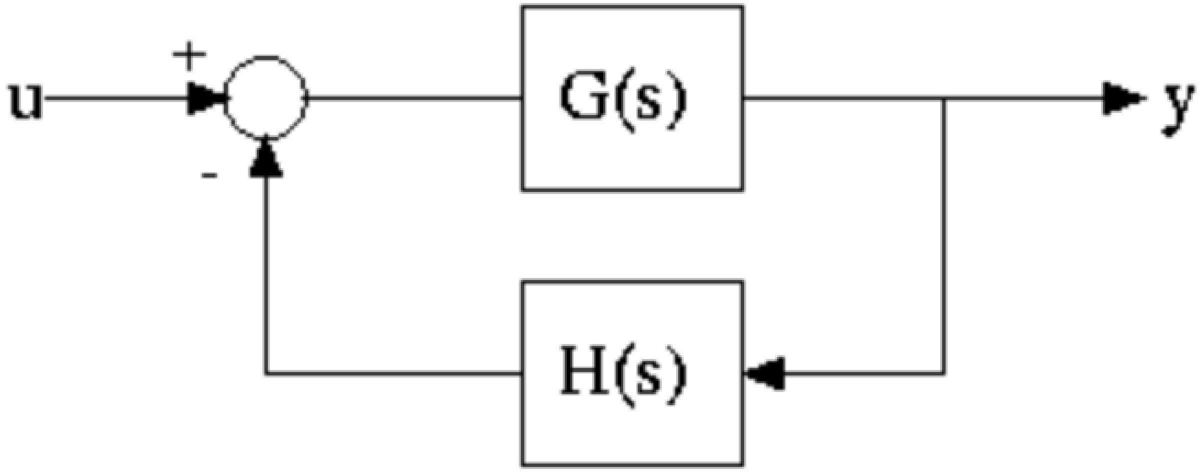
$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\dot{\mathbf{z}} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}u$$

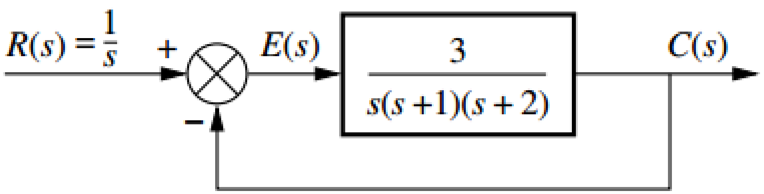
$$y = \begin{bmatrix} 5 & -1 \end{bmatrix}\mathbf{z}$$

System Control

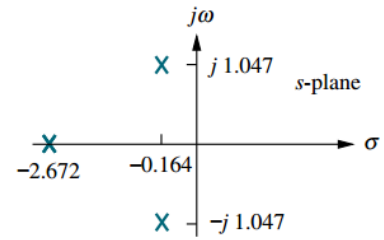
Stability



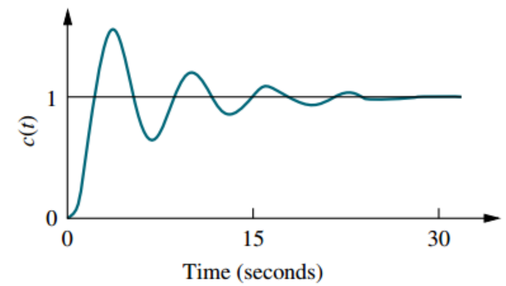
Stable & Unstable

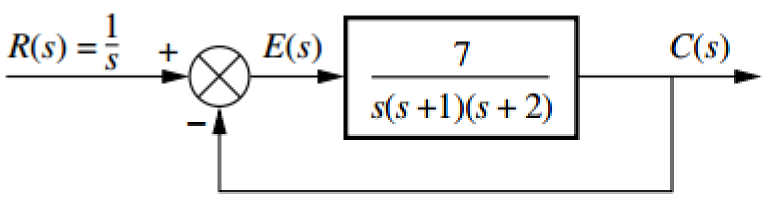


Stable system

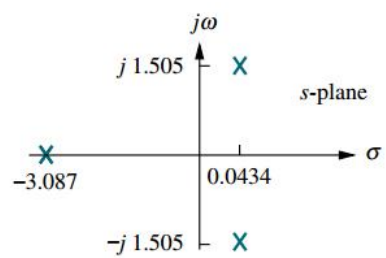


Stable system's closed-loop poles (not to scale)

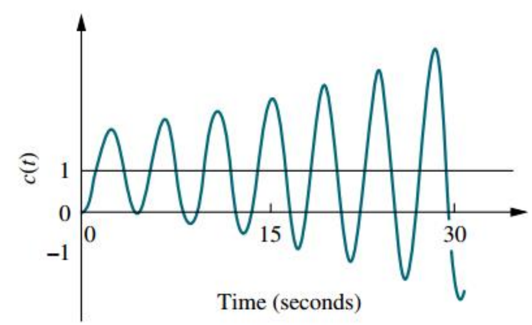




Unstable system

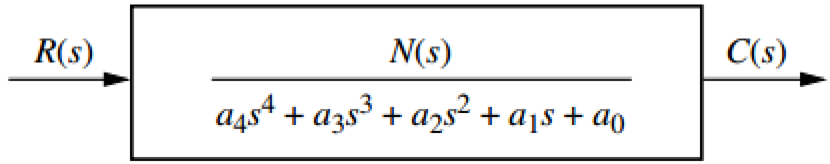


Unstable system's closed-loop poles (not to scale)



Routh-Hurwitz criterion

How many system poles are in the left half-plane, in the right half-plane, and on the imaginary axis?



s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\dfrac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

# of sign changes in the first column = # of roots in RHP

Routh-Hurwitz criterion: special cases (1)

Zero Only in the First Column

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

Stability via Epsilon Method

Label	First column	$\epsilon = +$	$\epsilon = -$
s^5	1	+	+
s^4	2	+	+
s^3	~~0~~ ϵ	+	−
s^2	$\frac{6\epsilon - 7}{\epsilon}$	−	+
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	+	+
s^0	3	+	+

Stability via Reverse Coefficients

$$s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0 \xrightarrow{\text{If } s \text{ is replaced by } 1/d,} \left(\frac{1}{d}\right)^n [1 + a_{n-1}d + \cdots + a_1 d^{(n-1)} + a_0 d^n] = 0$$

Routh-Hurwitz criterion: special cases (2)

Entire Row is Zero

When an entire row of the R-H array is zero, this indicates that there are complex conjugate pairs of roots that are mirror image of each other with respect to the real axis.

$$a(s) = s^5 + 5s^4 + 11s^3 + 23s^2 + 28s + 12$$

s^5	1	11	28	
s^4	5	23	12	
s^3	6.4	25.6		
s^2	3	12		
s^1	0	0		$\leftarrow a_1(s) = 3s^2 + 12$
New				
s	6	0		$\leftarrow \frac{da_1(s)}{ds} = 6s$
s^0	12			

No change in sign ➔ all roots except two are in LHP.

$$a_1(s) = 3(s^2+4) = 0$$

➔ $s = \pm j2$

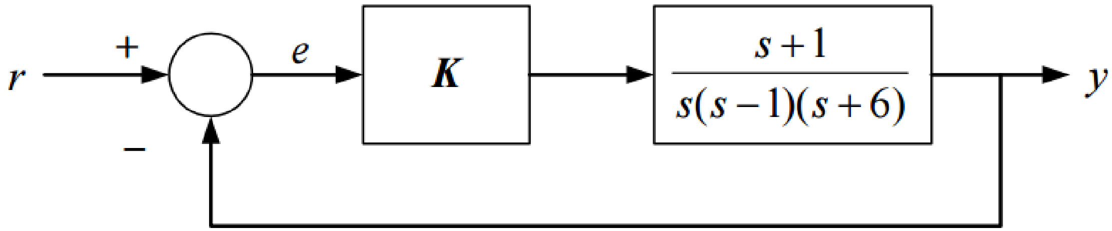
What if we replace 0 with ε.

If $\varepsilon > 0$, then no sign changes.

If $\varepsilon < 0$, then two sign changes.

➔ If $\varepsilon = 0$, there are two poles on the imaginary axis.

Stability design via Routh-Hurwitz (1)



C.E. $1+K\dfrac{s+1}{s(s-1)(s+6)}=0$

or $s^3+5s^2+(K-6)s+K=0$

R-H array

s^3	1	$K-6$
s^2	5	K
s^1	$(4K-30)/5$	
s^0	K	

For the system to remain stable, it is necessary that

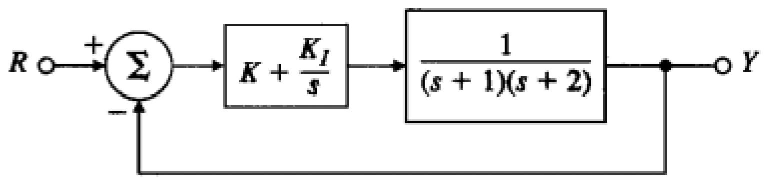
$$\frac{4K-30}{5}>0 \text{ and } K>0$$

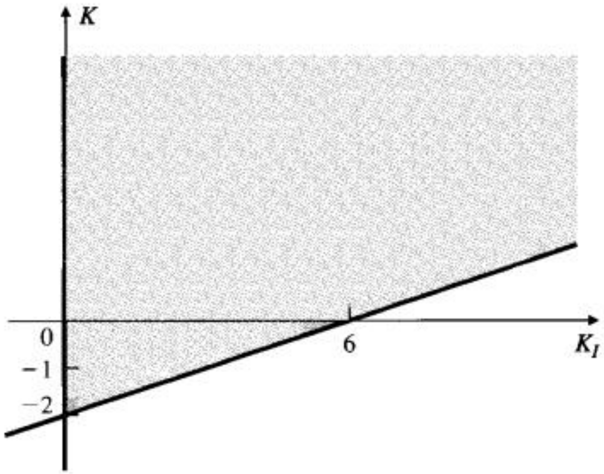
or $K>7.5$ and $K>0$

$$\Rightarrow \quad K>7.5$$

Closed-loop system $\dfrac{Y(s)}{R(s)}=\dfrac{K(s+1)}{s^3+5s^2+(K-6)s+K}$

Stability design via Routh-Hurwitz (2)

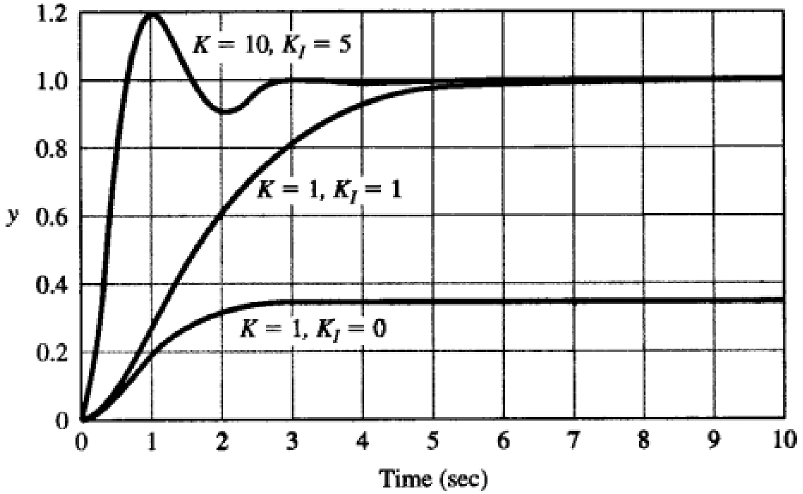




$$1+\left(K+\frac{K_I}{s}\right)\frac{1}{(s+1)(s+2)}=0, \qquad s^3+3s^2+(2+K)s+K_I=0.$$

The corresponding Routh array is

$$\begin{array}{lll} s^3: & 1 & 2+K \\ s^2: & 3 & K_I \\ s: & (6+3K-K_I)/3 & \\ s^0: & K_I & \end{array}$$



For asymptotic stability we must have

$$K_I > 0 \quad \text{and} \quad K > \frac{1}{3}K_I - 2.$$