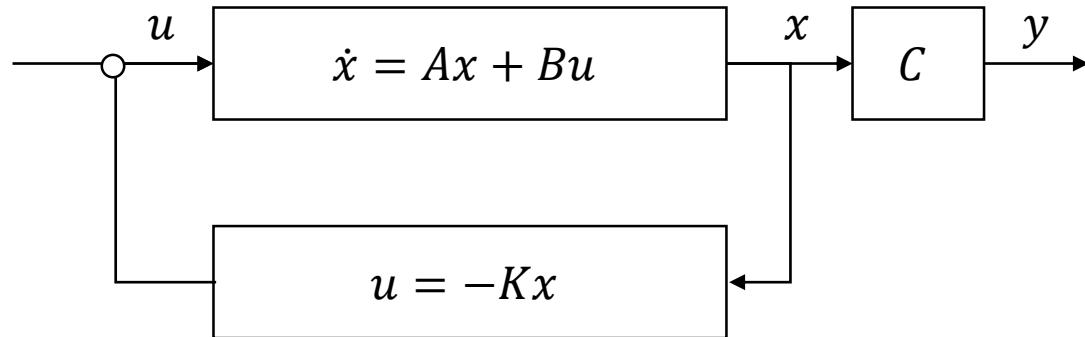


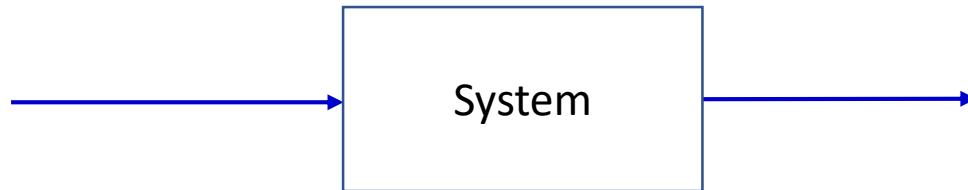
# Modern Control Theory

## state-space representation

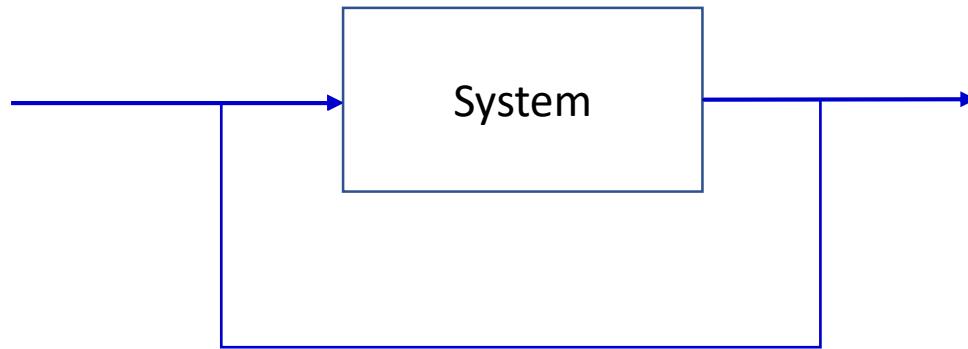


# *Open-loop (OL) & Closed-loop (CL)*

Open-loop control

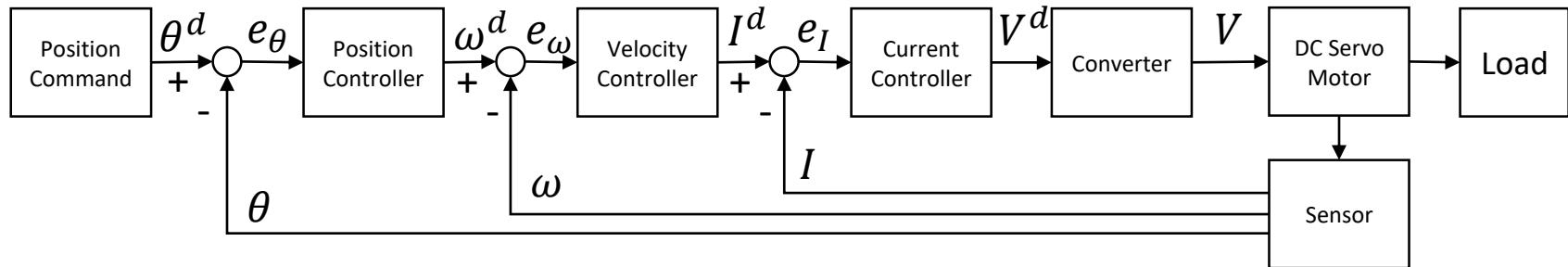


Closed-loop control

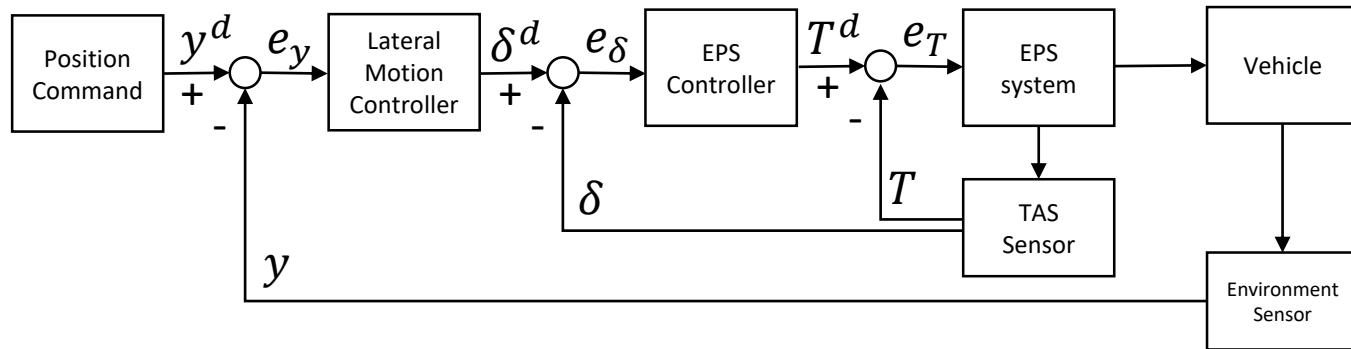


# *Example of closed-loop system*

Motor: position control (ex: Robot Arm)



Vehicle: lateral motion control (ex: Lane Keeping System)



# *What is state-space model?*

---

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \quad \xleftarrow{\text{state equation}}$$

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du} \quad \xleftarrow{\text{output equation}}$$

$\mathbf{x}$  = state vector

$\dot{\mathbf{x}}$  = derivative of the state vector with respect to time

$\mathbf{y}$  = output vector

$\mathbf{u}$  = input or control vector

$\mathbf{A}$  = system matrix

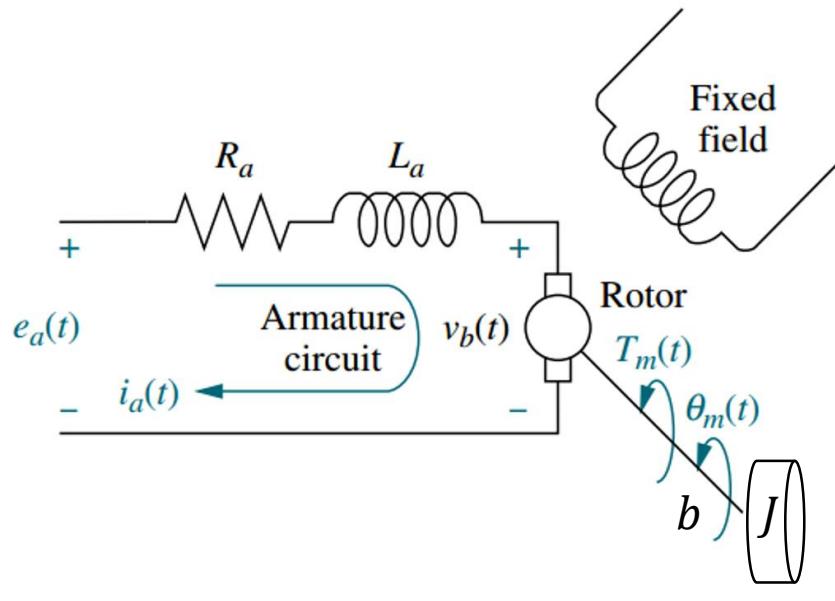
$\mathbf{B}$  = input matrix

$\mathbf{C}$  = output matrix

$\mathbf{D}$  = feedforward matrix

# DC motor

## Motor speed control



$$T = K_t i$$

$$v_b = K_b \dot{\theta}$$

- Newton's 2<sup>nd</sup> law

$$J \frac{d^2\theta}{dt^2} + b \frac{d\theta}{dt} = K_t i$$

- Kirchhoff's voltage law

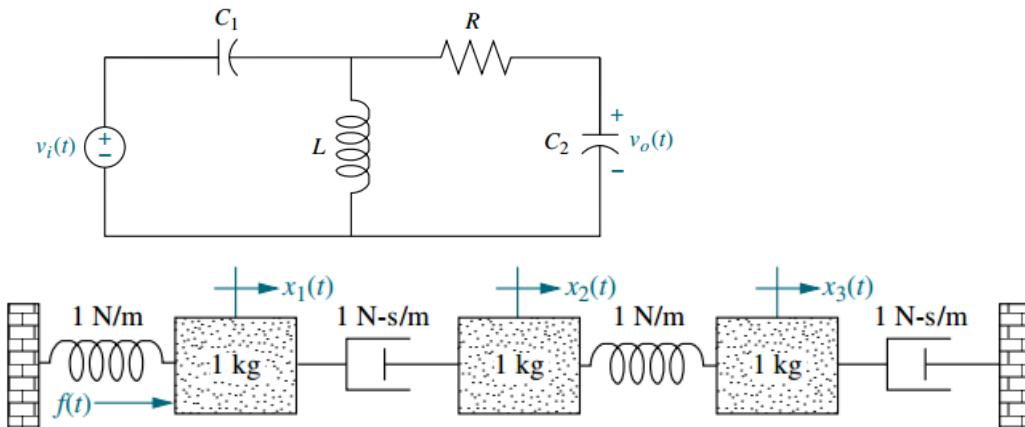
$$L \frac{di}{dt} + Ri = E_a - K_b \dot{\theta}$$

## How to choose state variables?

- What's our goal?
- Which values are input & output?

# State variable

- Minimum number of state variables

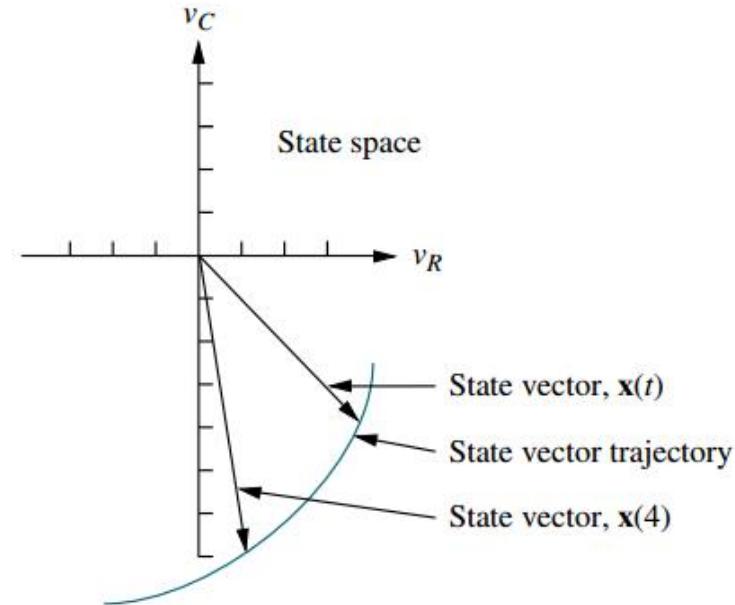


a second order system can be defined by two or more state variables

- Linear independence

$$S = K_n x_n + K_{n-1} x_{n-1} + \cdots + K_1 x_1$$

if their linear combination,  $S$ , equals zero *only* if every  $K_i = 0$  and *no*  $x_i = 0$  for all  $t \geq 0$ .

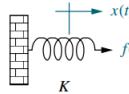
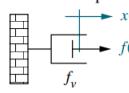


# State variable

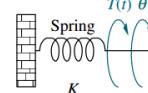
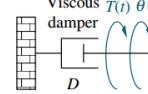
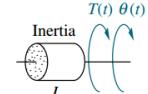
**TABLE 2.3** Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
	$v(t) = \frac{1}{C} \int_0^t i(\tau) d\tau$	$i(t) = C \frac{dv(t)}{dt}$	$v(t) = \frac{1}{C} q(t)$	$\frac{1}{Cs}$	$Cs$
	$v(t) = Ri(t)$	$i(t) = \frac{1}{R} v(t)$	$v(t) = R \frac{dq(t)}{dt}$	$R$	$\frac{1}{R} = G$
	$v(t) = L \frac{di(t)}{dt}$	$i(t) = \frac{1}{L} \int_0^t v(\tau) d\tau$	$v(t) = L \frac{d^2q(t)}{dt^2}$	$Ls$	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book:  $v(t)$  – V (volts),  $i(t)$  – A (amps),  $q(t)$  – Q (coulombs),  $C$  – F (farads),  $R$  –  $\Omega$  (ohms),  $G$  –  $\Omega$  (mhos),  $L$  – H (henries).

Component	Force-velocity	Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
	$f(t) = K \int_0^t v(\tau) d\tau$	$f(t) = Kx(t)$	$K$
	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
	$f(t) = M \frac{dv(t)}{dt}$	$f(t) = M \frac{d^2x(t)}{dt^2}$	$M s^2$

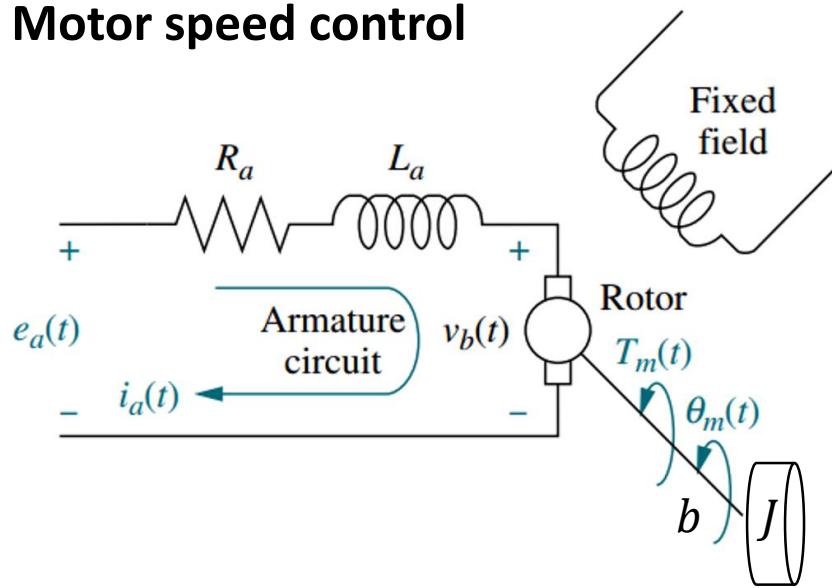
Note: The following set of symbols and units is used throughout this book:  $f(t)$  = N (newtons),  $x(t)$  = m (meters),  $v(t)$  = m/s (meters/second),  $K$  = N/m (newtons/meter),  $f_v$  = N-s/m (newton-seconds/meter),  $M$  = kg (kilograms = newton-seconds<sup>2</sup>/meter).

Component	Torque-angular velocity	Torque-angular displacement	Impedance $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K \int_0^t \omega(\tau) d\tau$	$T(t) = K\theta(t)$	$K$
	$T(t) = D\omega(t)$	$T(t) = D \frac{d\theta(t)}{dt}$	$Ds$
	$T(t) = J \frac{d\omega(t)}{dt}$	$T(t) = J \frac{d^2\theta(t)}{dt^2}$	$J s^2$

Note: The following set of symbols and units is used throughout this book:  $T(t)$  – N-m (newton-meters),  $\theta(t)$  – rad(radians),  $\omega(t)$  – rad/s(radians/second),  $K$  – N-m/rad(newton-meters/radian),  $D$  – N-m-s/rad (newton-meters-seconds/radian).  $J$  – kg-m<sup>2</sup>(kilograms-meters<sup>2</sup> – newton-meters-seconds<sup>2</sup>/radian).

# State-space model: DC motor

## Motor speed control



- Newton's 2<sup>nd</sup> law

$$J \frac{d^2\theta}{dt^2} + b \frac{d\theta}{dt} = K_t i$$

- Kirchhoff's voltage law

$$L \frac{di}{dt} + Ri = E_a - K_b \dot{\theta}$$

- State equation

$$\mathbf{x} = [\dot{\theta} \quad i]^T$$

$$\mathbf{y} = \dot{\theta}$$

$$\mathbf{u} = E_a$$

- Output equation

$$\dot{\mathbf{x}} = \begin{bmatrix} -\frac{b}{J} & \frac{K_t}{J} \\ -\frac{K_b}{L} & -\frac{R}{L} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = [1 \quad 0] \mathbf{x}$$

# *State-space model: DC motor*

## Motor **speed** control

$$\mathbf{x} = [\dot{\theta} \quad i]^T$$

$$\mathbf{y} = \dot{\theta}$$

$$\mathbf{u} = E_a$$

- State equation

$$\dot{\mathbf{x}} = \begin{bmatrix} -\frac{b}{J} & \frac{K_t}{J} \\ -\frac{K_b}{L} & -\frac{R}{L} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} \mathbf{u}$$

- Output equation

$$\mathbf{y} = [1 \quad 0] \mathbf{x}$$

## Motor **position** control

$$\mathbf{x} = [\dot{\theta} \quad \theta \quad i]^T$$

$$\mathbf{y} = \theta$$

$$\mathbf{u} = E_a$$

- State equation

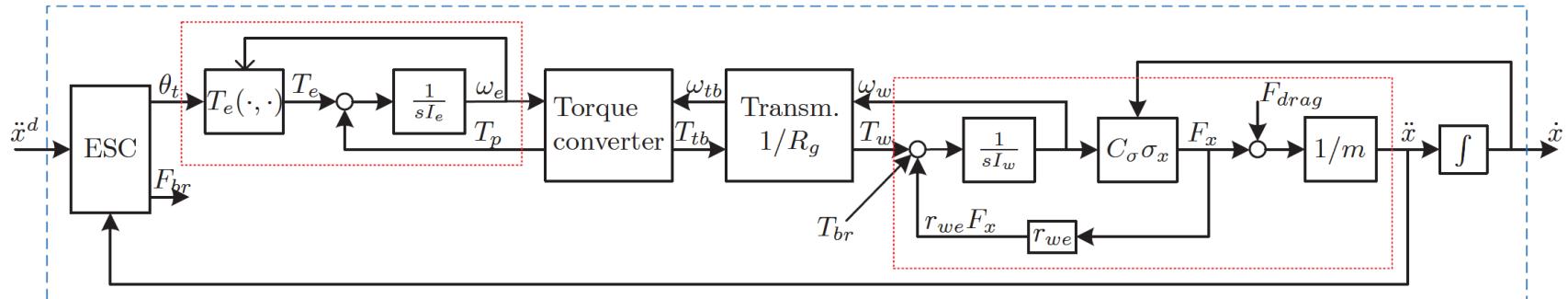
$$\dot{\mathbf{x}} = \begin{bmatrix} -\frac{b}{J} & 0 & \frac{K_t}{J} \\ \frac{1}{L} & 0 & 0 \\ -\frac{K_b}{L} & 0 & -\frac{R}{L} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1/L \end{bmatrix} \mathbf{u}$$

- Output equation

$$\mathbf{y} = [0 \quad 1 \quad 0] \mathbf{x}$$

# *Longitudinal vehicle model*

## Vehicle longitudinal control



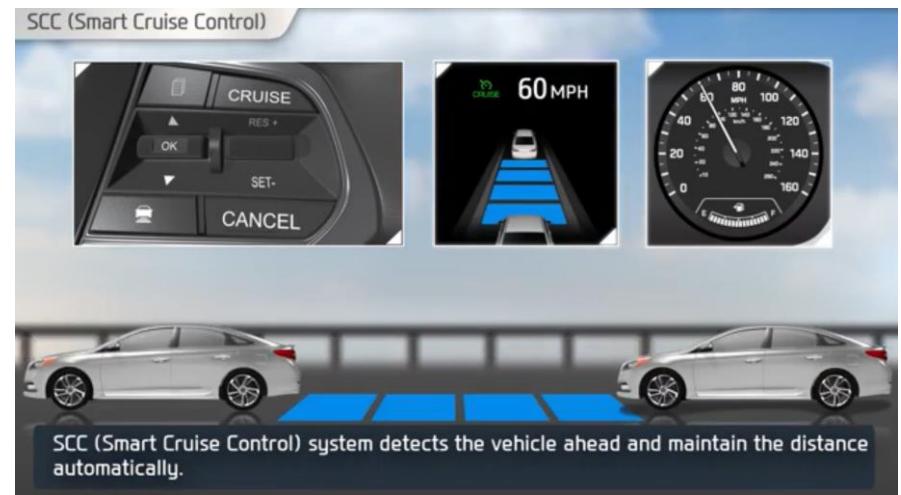
## - Input & output

$$u = \ddot{x}^d$$

$$y = \dot{x}$$

### - Simplified transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s(1 + \tau s)}$$



[출처: 현대자동차]

# Transfer function to state-space model

## ► Transfer function

$$\frac{Y(s)}{U(s)} = \frac{1}{s(1 + \tau s)}$$



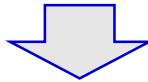
$$s(1 + \tau s)Y(s) = U(s)$$

$$sY(s) + \tau s^2 Y(s) = U(s)$$



Laplace transform

$$\dot{y}(t) + \tau \ddot{y}(t) = u(t)$$



$$u = \ddot{x}^d \quad y = \dot{x}$$

$$a + \tau \dot{a} = a^d$$

## ► State-space

$$\mathbf{x} = [v \quad a]^T$$

$$\mathbf{y} = v$$

$$\mathbf{u} = a^d$$

- State equation

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & -1/\tau \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1/\tau \end{bmatrix} \mathbf{u}$$

- Output equation

$$\mathbf{y} = [1 \quad 0] \mathbf{x}$$

# *State-space model to transfer function*

---

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Bu} \\ \mathbf{y} &= \mathbf{Cx} + \mathbf{Du}\end{aligned}\quad \longrightarrow \quad T(s) = Y(s)/U(s)$$

Laplace transform

$$s\mathbf{X}(s) = \mathbf{AX}(s) + \mathbf{BU}(s)$$

$$\mathbf{Y}(s) = \mathbf{CX}(s) + \mathbf{DU}(s)$$

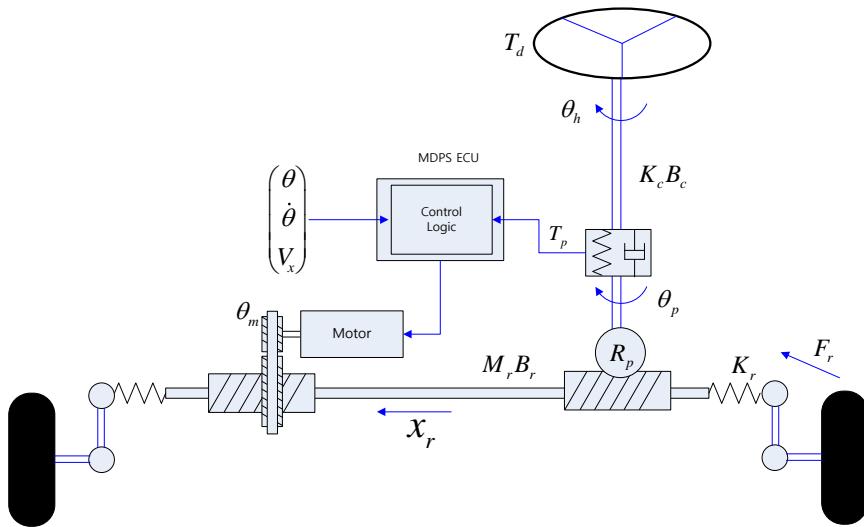
Solving for  $\mathbf{X}(s)$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{BU}(s)$$

$$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{BU}(s) + \mathbf{DU}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} + \mathbf{D}] \mathbf{U}(s)$$

$$T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} + \mathbf{D}$$

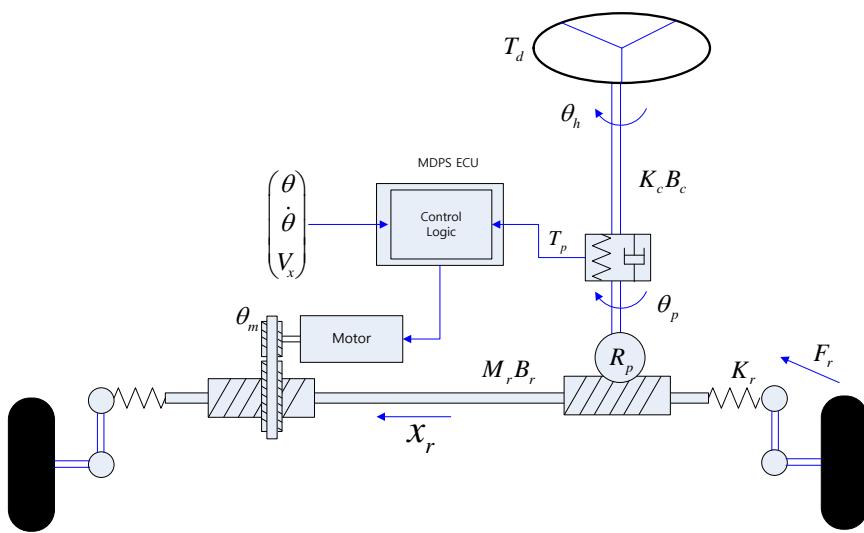
# Electric power steering



기호	설명
	Steering wheel angle
	Motor angle
	Pinion angle
	Rack position
	Motor current
	Driver torque
	Steering column torque
	Friction Torque
	Road reaction torque on the rack and pinion

기호	설명
	Motor moment of inertia
	Motor shaft viscous damping
	Motor torque, voltage constant
	Steering column moment of inertia
	Steering column viscous damping
	Steering column stiffness
	Mass of the rack
	Viscous damping of the rack
	Tire spring rate
	Steering column pinion radius
	Motor gear ratio
	Spring constant
	Pinion load
	Motor load

# Electric power steering



Newton 2<sup>nd</sup> eq. for steering wheel

$$J_h \ddot{\theta}_h + B_h \dot{\theta}_h + K_h (\theta_h - \theta_p) = T_d$$

Newton 2<sup>nd</sup> eq. for motor and rack pinion

$$J_m \ddot{\theta}_m + B_m \dot{\theta}_m = K_t I_m - \tau_l$$

$$M_r \ddot{x}_r + B_r \dot{x}_r + K_r x_r = \frac{\tau_p}{R_p} - F_r + \frac{N \tau_l}{R_m}$$

$$x_r = \frac{R_m \theta_m}{N} = R_p \theta_p$$

$$J_m \frac{N}{R_m} R_p \ddot{\theta}_p + B_m \frac{N}{R_m} R_p \dot{\theta}_p = K_t I_m - \tau_l$$

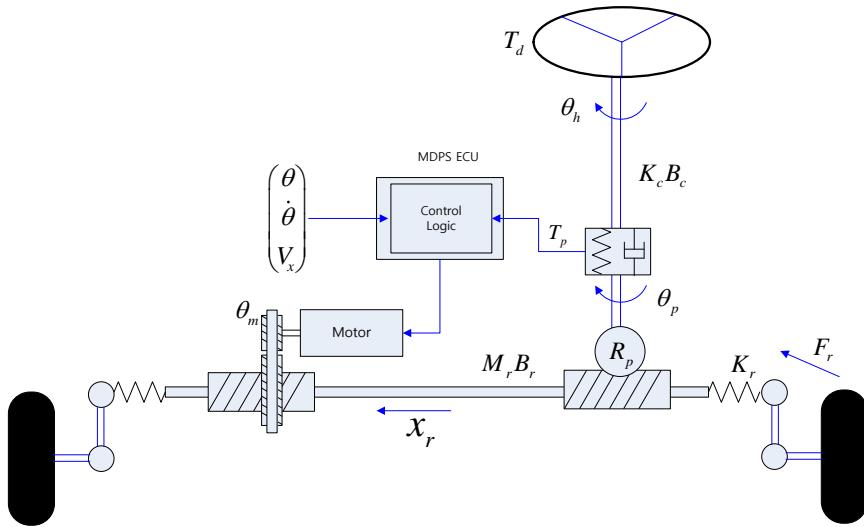
$$M_r R_p \ddot{\theta}_p + B_r R_p \dot{\theta}_p + K_r R_p \theta_p = \frac{\tau_p}{R_p} - F_r + \frac{N \tau_l}{R_m}.$$

$$\tau_p = K_h (\theta_h - \theta_p)$$

$$\underbrace{\left( \frac{J_m R_p N^2}{R_m^2} + M_r R_p \right)}_{J_{eq}} \ddot{\theta}_p + \underbrace{\left( \frac{B_m R_p N^2}{R_m^2} + B_r R_p \right)}_{B_{eq}} \dot{\theta}_p + \left( K_r R_p - \frac{K_h}{R_p} \right) \theta_p - \frac{K_h}{R_p} \theta_h = -F_r + \frac{K_t N}{R_m} I_m$$

# Electric power steering

4<sup>th</sup> order modeling



$$x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} = \begin{bmatrix} \theta_h & \dot{\theta}_h & \theta_p & \dot{\theta}_p \end{bmatrix}^T$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{K_h}{J_h} x_1 - \frac{B_h}{J_h} x_2 + \frac{K_h}{J_h} x_3 + \frac{1}{J_h} T_d$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = \frac{K_h}{R_p J_{eq}} x_1 - \left( \frac{K_h R_p}{J_{eq}} - \frac{K_h}{J_{eq} R_p} \right) x_3 - \frac{B_{eq}}{J_{eq}} x_4 - \frac{1}{J_{eq}} F_r + \frac{K_t N}{R_m J_{eq}} I_m$$

# Summary

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- Open-loop & Closed-loop
- State-space representation
  - DC motor, vehicle, electric power steering

