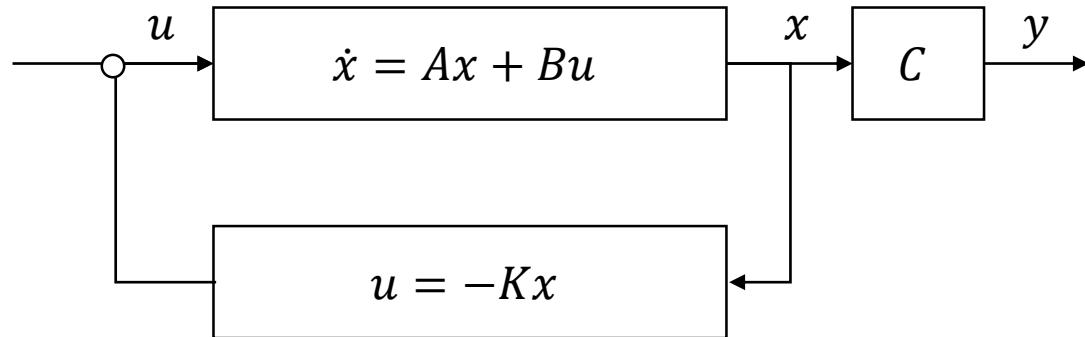


# Modern Control Theory

## Full state feedback



# Solution

## Motor speed control

- State equation                            - Output equation

$$\begin{aligned} \mathbf{x} &= [\dot{\theta} \quad i]^T \\ \mathbf{y} &= \dot{\theta} \\ \mathbf{u} &= E_a \\ x(0) &= [1 \quad 1]^T \end{aligned}$$
$$\dot{\mathbf{x}} = \begin{bmatrix} -\frac{b}{J} & \frac{K_t}{J} \\ -\frac{K_b}{L} & -\frac{R}{L} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} \mathbf{u}$$
$$\mathbf{y} = [1 \quad 1] \mathbf{x}$$

$$J = 4, b = 2, K_t = 4, K_b = 1, R = 3, L = 0.1$$

1. Calculate the eigenvector and eigenvalue for system matrix  $\mathbf{A}$
2. Find state equation solution for the given model

$$A = \begin{bmatrix} 1 \\ -\frac{1}{2} & 1 \\ -10 & -30 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 10 \end{bmatrix} \quad C = [1 \quad 1] \quad D = [0]$$

$$(sI - A)^{-1} = \begin{bmatrix} s + \frac{1}{2} & -1 \\ 10 & s + 30 \end{bmatrix}^{-1} = \frac{1}{\left(s + \frac{1}{2}\right)(s + 30) + 10} \begin{bmatrix} s + 30 & 1 \\ -10 & s + \frac{1}{2} \end{bmatrix}$$

# Solution

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$$(sI - A)^{-1} = \begin{bmatrix} s + \frac{1}{2} & -1 \\ 10 & s + 30 \end{bmatrix}^{-1} = \frac{1}{\left(s + \frac{1}{2}\right)(s + 30) + 10} \begin{bmatrix} s + 30 & 1 \\ -10 & s + \frac{1}{2} \end{bmatrix}$$

$$\det(sI - A) = s^2 + 30.5s + 25$$

$$s = \lambda = -29.657, -0.843$$

$$\begin{bmatrix} -\frac{1}{2} & 1 \\ -10 & -30 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -29.657 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{aligned} -\frac{1}{2}x_1 + x_2 &= -29.657x_1 \\ -10x_1 - 30x_2 &= -29.657x_2 \end{aligned} \quad \begin{aligned} 29.157x_1 &= -x_2 \\ x_1 &= -0.0343x_2 \end{aligned} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -0.0343c \\ c \end{bmatrix}$$

# Solution

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$$(sI - A)^{-1} = \begin{bmatrix} s + \frac{1}{2} & -1 \\ 10 & s + 30 \end{bmatrix}^{-1} = \frac{1}{\left(s + \frac{1}{2}\right)(s + 30) + 10} \begin{bmatrix} s + 30 & 1 \\ -10 & s + \frac{1}{2} \end{bmatrix}$$

$$\det(sI - A) = s^2 + 30.5s + 25$$

$$s = \lambda = -29.657, \color{red}{-0.843}$$

$$\begin{bmatrix} -\frac{1}{2} & 1 \\ -10 & -30 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -0.843 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{aligned} -\frac{1}{2}x_1 + x_2 &= -0.843x_1 \\ -10x_1 - 30x_2 &= -0.843x_2 \end{aligned}$$

$$\begin{aligned} 0.343x_1 &= -x_2 \\ x_1 &= -2.9157x_2 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2.9157c \\ c \end{bmatrix}$$

# Solution

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau) d\tau$$

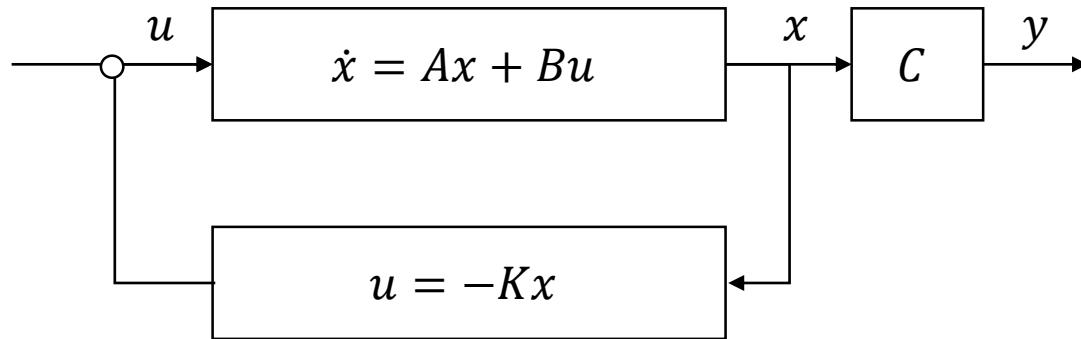
**Zero-input response**

$$\exp(At) = \mathcal{L}^{-1} \begin{pmatrix} \frac{s+30}{s^2 + 30.5s + 25} & \frac{1}{s^2 + 30.5s + 25} \\ \frac{-10}{s^2 + 30.5s + 25} & \frac{s + \frac{1}{2}}{s^2 + 30.5s + 25} \end{pmatrix}, \quad e^{At}x_0 = \mathcal{L}^{-1} \begin{pmatrix} \frac{s+30}{s^2 + 30.5s + 25} \\ \frac{-10}{s^2 + 30.5s + 25} \end{pmatrix}$$

**Zero-state response**

$$\begin{aligned} & A^{-1} (\exp(At) - I)B \\ &= \begin{bmatrix} -1.2 & -0.04 \\ 0.4 & -0.02 \end{bmatrix} \mathcal{L}^{-1} \begin{pmatrix} \frac{s+30}{s^2 + 30.5s + 25} & \frac{1}{s^2 + 30.5s + 25} \\ \frac{-10}{s^2 + 30.5s + 25} & \frac{s + \frac{1}{2}}{s^2 + 30.5s + 25} \end{pmatrix} \begin{bmatrix} 0 \\ 10 \end{bmatrix} \end{aligned}$$

# State feedback



$$u(t) = -Kx(t)$$

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$



$$\dot{x}(t) = Ax(t) + BKx(t)$$

$$= (A + BK)x(t)$$

**K** should be chosen such that  $(A + BK)$  is Hurwitz!

# *State feedback*

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Assume that the single-input system dynamics are given by

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{y}(t) &= C\mathbf{x}(t)\end{aligned}$$

Assume a full-state feedback of the form:

$$\mathbf{u}(t) = \mathbf{r} - K\mathbf{x}(t)$$

where  $\mathbf{r}$  is some **reference input** and the **gain**  $K$  is  $\mathbb{R}^{1 \times n}$

- If  $\mathbf{r} = 0$ , we call this controller a **regulator**

# Pole-placement

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## Objective

Design K so that the system has the desired properties

- Closed-loop stable
- Put desired poles

Consider:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \quad \text{put the poles at } s = -5, -6$$

# *Pole-placement*

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Consider this system:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

# *Reference tracking*

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$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + Bu \quad y = C\mathbf{x}(t)$$

$$u = r - K\mathbf{x}(t)$$

For **good tracking performance** we want

$$y(t) \approx r(t) \text{ as } t \rightarrow \infty \quad \longleftrightarrow \quad \lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sY(s)$$



$$sY(s) \approx sR(s) \text{ as } s \rightarrow 0 \quad \Rightarrow \quad \left. \frac{Y(s)}{R(s)} \right|_{s=0} = 1$$

# *Reference tracking*

---

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t)\end{aligned}$$

- Already designed  $K = [14 \ 57]$  so the closed-loop system is

$$\begin{aligned}\dot{\mathbf{x}}(t) &= (A - BK)\mathbf{x}(t) + Br \\ y &= C\mathbf{x}(t)\end{aligned}$$

which gives the transfer function

$$\begin{aligned}\frac{Y(s)}{R(s)} &= C(sI - (A - BK))^{-1} B \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s + 13 & 56 \\ -1 & s - 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{s - 2}{s^2 + 11s + 30}\end{aligned}$$

# *Reference tracking*

Using extra gain to scale the closed-loop TF

$$u = \bar{N}r - K\mathbf{x}(t)$$



$$\begin{aligned}\dot{\mathbf{x}}(t) &= (A - BK)\mathbf{x}(t) + B\bar{N}r \\ y &= C\mathbf{x}(t)\end{aligned}$$

$$\frac{Y(s)}{R(s)} = C(sI - (A - BK))^{-1} B \bar{N} = G_{cl}(s) \bar{N}$$



$$\bar{N} = G_{cl}(0)^{-1} = - (C(A - BK)^{-1}B)^{-1}$$

# *MATLAB code*

```
clear all                                %% controllable check
close all
clc

A = [1 1;                               % CM = [B A*B];
      1 2];                           % inv(CM);
% A = [1 1;                           % rank(CM);
%       0 2];

B = [1;0];                                Co = ctrb(A,B);
% B = [0;1];                           inv(Co);
                                         rank(Co);
                                         size(A,1) == rank(Co);

C = [1 0];                                %% design controller
D = [0];                                  P = [-5 -6];
                                         K = place(A,B,P);      % u = - K x

%% stability check
eig(A);                                 Acl = A-B*K;
                                         eig(Acl);

                                         sys1 = ss(A,B,C,D);
                                         sys2 = ss(Acl,-15*B,C,D);
                                         [y1,t1] = step(sys1);
                                         [y2,t2] = step(sys2);

                                         figure
                                         plot(y1,t1)
                                         figure
                                         plot(y2,t2)
```