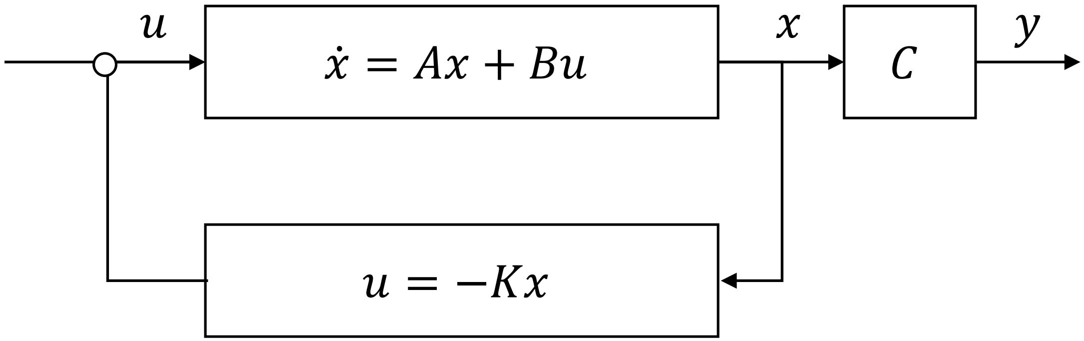
Modern Control Theory

State observer & output feedback



State feedback design

State feedback

$$\mathrm{u}(t) = -K\mathrm{x}(t)$$

$$\begin{aligned}\dot{\mathrm{x}}(t) &= A\mathrm{x}(t) + B\mathrm{u}(t) \\ \mathrm{y}(t) &= C\mathrm{x}(t) + D\mathrm{u}(t)\end{aligned} \quad \Longrightarrow \quad \begin{aligned}\dot{\mathrm{x}}(t) &= A\mathrm{x}(t) - BK\mathrm{x}(t) \\ &= (A - BK)\mathrm{x}(t)\end{aligned}$$

Full-state feedback / Pole placement

- Check Stability
- Check Controllability
- Design Pole

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \quad \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \quad \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Stabilizable

How can we handle uncontrollable system?

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

Controllable & Stabilizable

Controllable

- **Definition:** An LTI **system** is **controllable** if, for every $\mathbf{x}^{\star}(t)$ and every finite $T > 0$, there exists an input function $\mathbf{u}(t)$, $0 < t \leq T$, such that the system state goes from $\mathbf{x}(0) = 0$ to $\mathbf{x}(T) = \mathbf{x}^{\star}$.

Stabilizable

The system is stabilizable if there exists matrix K such that A + BK (or A-BK) is a stability matrix.

State observer design

For given system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \qquad \mathbf{y} = C\mathbf{x} = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}$$

We want design the state feedback as follows

$$u = -K\mathbf{x} = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

But, the first state is not available

State observer design

Let us consider the following system

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu \qquad \mathbf{y} = C\mathbf{x}$$

Let us define estimated state $\hat{\mathbf{x}}$, then observer is modeled as

$$\dot{\hat{\mathbf{x}}} = A\hat{\mathbf{x}} + Bu + L(\mathbf{y} - \hat{\mathbf{y}}) \qquad \hat{\mathbf{y}} = C\hat{\mathbf{x}}$$

Define the estimation error $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$

$$\begin{aligned} \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} &= A(\mathbf{x} - \hat{\mathbf{x}}) - LC(\mathbf{x} - \hat{\mathbf{x}}) \\ &= (A - LC)(\mathbf{x} - \hat{\mathbf{x}}) \end{aligned}$$

The error converges to zero if $(A - LC)$ is Hurwitz

Observable

- **Definition:** An LTI **system** is **observable** if the initial state $\mathbf{x}(0)$ can be **uniquely deduced** from the knowledge of the input $\mathbf{u}(t)$ and output $\mathbf{y}(t)$ for all t between 0 and any finite $T > 0$.
  - If $\mathbf{x}(0)$ can be deduced, then we can reconstruct $\mathbf{x}(t)$ exactly because we know $\mathbf{u}(t) \Rrightarrow$ we can find $\mathbf{x}(t)\ \forall\ t$.

Duality

The dual of the Linear Time Invariant system is given by

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu \qquad \Longleftrightarrow \qquad \dot{\mathbf{z}} = A^T\mathbf{z} + C^T v$$
$$\mathbf{y} = C\mathbf{x} \qquad\qquad\qquad \mathbf{w} = B^T\mathbf{z}$$

- Controllable matrix

- Observable matrix

Controller

$$\dot{\mathbf{x}} = (A - BK)\mathbf{x}$$

$$A, B, K$$

Observer

$$\dot{\mathbf{e}} = (A - LC)\mathbf{e}$$

$$A^T, C^T, L^T$$

State feedback

State feedback

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu \qquad u = Fr - K\mathbf{x}$$
$$\mathbf{y} = C\mathbf{x}$$

Pole-placement
Optimal control (LQ)
H infinite FSFB

State observer

$$\dot{\hat{\mathbf{x}}} = A\hat{\mathbf{x}} + Bu + L(\mathbf{y} - \hat{\mathbf{y}})$$

Luenberger observer
Kalman filter
H infinite filter

Output feedback

$$\mathbf{x} = [\mathbf{x} \quad \tilde{\mathbf{x}}]^T$$

LQG / H infinite control

State feedback