Modern Control Theory
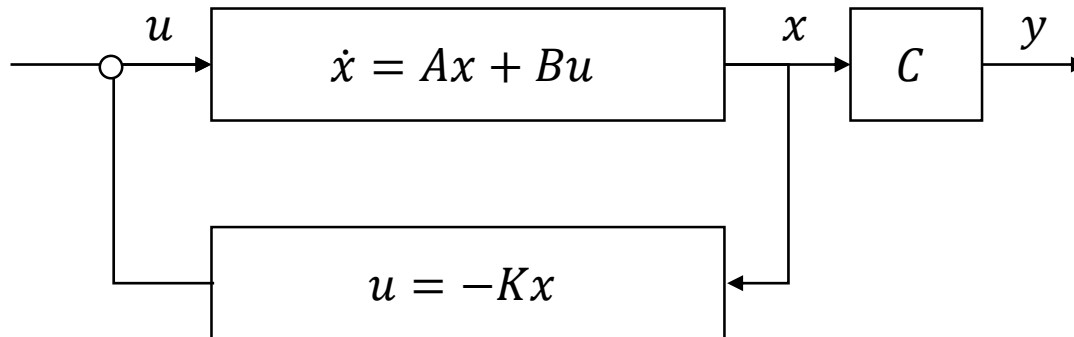## Linear Quadratic Control

Review

Full-state feedback

$$\dot{\mathbf{x}} = (A + BK)\mathbf{x}$$

$$u = K\mathbf{x}$$

Luenberger observer

$$\dot{\hat{\mathbf{x}}} = A\hat{\mathbf{x}} + Bu + L(C\mathbf{x} - C\hat{\mathbf{x}})$$

How to choose the gain *K & L* ?

We want to stabilize

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0.$$

and minimize

$$J := \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t)\, dt$$

인천대학교
INCHEON NATIONAL UNIVERSITY

Some stability definitions

we consider nonlinear time-invariant system $\dot{x} = f(x)$, where $f : \mathbf{R}^n \to \mathbf{R}^n$

a point $x_e \in \mathbf{R}^n$ is an *equilibrium point* of the system if $f(x_e) = 0$

x_e is an equilibrium point $\iff x(t) = x_e$ is a trajectory

suppose x_e is an equilibrium point

- system is *globally asymptotically stable* (G.A.S.) if for every trajectory $x(t)$, we have $x(t) \to x_e$ as $t \to \infty$
  (implies x_e is the unique equilibrium point)

- system is *locally asymptotically stable* (L.A.S.) near or at x_e if there is an $R > 0$ s.t. $\|x(0) - x_e\| \leq R \implies x(t) \to x_e$ as $t \to \infty$

Energy and dissipation functions

consider nonlinear system $\dot{x} = f(x)$, and function $V : \mathbf{R}^n \to \mathbf{R}$

we define $\dot{V} : \mathbf{R}^n \to \mathbf{R}$ as $\dot{V}(z) = \nabla V(z)^T f(z)$

$\dot{V}(z)$ gives $\dfrac{d}{dt} V(x(t))$ when $z = x(t)$, $\dot{x} = f(x)$

we can think of V as *generalized energy function*, and $-\dot{V}$ as the associated *generalized dissipation function*

Positive definite functions

a function $V : \mathbf{R}^n \to \mathbf{R}$ is *positive definite* (PD) if

- $V(z) \geq 0$ for all z
- $V(z) = 0$ if and only if $z = 0$
- all sublevel sets of V are bounded

last condition equivalent to $V(z) \to \infty$ as $z \to \infty$

example: $V(z) = z^T P z$, with $P = P^T$, is PD if and only if $P > 0$

Lyapunov theory

Lyapunov theory is used to make conclusions about trajectories of a system $\dot{x} = f(x)$ (*e.g.*, G.A.S.) *without finding the trajectories* (*i.e.*, solving the differential equation)

a typical Lyapunov theorem has the form:

- **if** there exists a function $V : \mathbf{R}^n \to \mathbf{R}$ that satisfies some conditions on V and $\dot{V}$

- **then**, trajectories of system satisfy some property

if such a function V exists we call it a *Lyapunov function* (that proves the property holds for the trajectories)

Lyapunov function V can be thought of as *generalized energy function* for system

Lyapunov equation

What is the Lyapunov theory??

the *Lyapunov equation* is

$$A^T P + PA + Q = 0$$

where $A, \ P, \ Q \in \mathbf{R}^{n \times n}$, and $P, \ Q$ are symmetric

interpretation: for linear system $\dot{x} = Ax$, if $V(z) = z^T P z$, then

$$\dot{V}(z) = (Az)^T P z + z^T P (Az) = -z^T Q z$$

i.e., if $z^T P z$ is the (generalized)*energy*, then $z^T Q z$ is the associated (generalized) *dissipation*

Lyapunov equation

we consider system $\dot{x} = Ax$, with $\lambda_1, \ldots, \lambda_n$ the eigenvalues of A

if $P > 0$, then

boundedness condition: if $P > 0$, $Q \geq 0$ then

- all trajectories of $\dot{x} = Ax$ are bounded
  (this means $\Re\lambda_i \leq 0$, and if $\Re\lambda_i = 0$, then λ_i corresponds to a Jordan block of size one)

- the ellipsoids $\{z \mid z^T P z \leq a\}$ are invariant

if $P > 0$, $Q > 0$ then the system $\dot{x} = Ax$ is (globally asymptotically) stable, *i.e.*, $\Re\lambda_i < 0$

Lyapunov equation

Unique solution

We assume $A \in \mathbf{R}^{n \times n}$, $P = P^T \in \mathbf{R}^{n \times n}$. It follows that $Q = Q^T \in \mathbf{R}^{n \times n}$.

Continuous-time linear system: for $\dot{x} = Ax$, $V(z) = z^T P z$, we have $\dot{V}(z) = -z^T Q z$, where P, Q satisfy (continuous-time) Lyapunov equation $A^T P + P A + Q = 0$.

thus if A is stable, for any Q there is exactly one solution P of Lyapunov equation $A^T P + P A + Q = 0$

$$P = \int_0^\infty e^{tA^T} Q e^{tA} \, dt$$

Lyapunov equation

$$P = \int_0^\infty e^{tA^T} Q e^{tA} \, dt$$

to see this, we note that

$$
\begin{aligned}
A^T P + P A &= \int_0^\infty \left(A^T e^{tA^T} Q e^{tA} + e^{tA^T} Q e^{tA} A \right) \, dt \\
&= \int_0^\infty \left(\frac{d}{dt} e^{tA^T} Q e^{tA} \right) \, dt \\
&= e^{tA^T} Q e^{tA} \Big|_0^\infty \\
&= -Q
\end{aligned}
$$

Lyapunov equation

1. $\dot{x} = x$

2. $\dot{x} = -x$

3. $\dot{x}_1 = -x_1$
   $\dot{x}_2 = -2x_2$

4. $\dot{x}_1 = x_1$
   $\dot{x}_2 = -2x_2$

5. $\dot{x}_1 = x_1 - x_2$
   $\dot{x}_2 = -2x_2$

Optimal control

How to choose the gain *K & L* ?

We want to stabilize

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0.$$

and minimize

$$J := \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t)\, dt$$

Assumptions

a) $Q \succeq 0,\ R \succ 0$;

b) (A, B) stabilizable;

인천대학교
INCHEON NATIONAL UNIVERSITY

Optimal control

$$u = K\mathbf{x}$$

$$J := \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t)\, dt$$

$$J = \int_0^\infty x(t)^T (Q + K^T R K) x(t)\, dt$$

$$\dot{x}(t) = A x(t) + B u(t), \quad x(0) = x_0.$$

$$\dot{x} = (A + BK)x, \quad x(0) = x_0.$$

for a given K and x_0

$$x(t) = e^{(A+BK)t} x_0.$$

and

$$J = \int_0^\infty x_0^T e^{(A+BK)^T t} (Q + K^T R K) e^{(A+BK)t} x_0\, dt$$

$$= x_0^T \left(\int_0^\infty e^{(A+BK)^T t} (Q + K^T R K) e^{(A+BK)t}\, dt \right) x_0.$$

Optimal control

$$J = x_0^T \left(\int_0^\infty e^{(A+BK)^T t}(Q + K^T RK)e^{(A+BK)t}\, dt \right) x_0.$$

J can be computed as

$$J = x_0^T X x_0$$

where X is the solution to the Lyapunov equation

$$(A + BK)^T X + X(A + BK) + Q + K^T RK = 0.$$

above equation can be rewriting in the form because

$$A^T X + XA - XBR^{-1}B^T X + Q + (XBR^{-1} + K^T)R(R^{-1}B^T X + K) = 0.$$

Optimal control

$$A^T X + XA - XBR^{-1}B^T X + Q + (XBR^{-1} + K^T)R(R^{-1}B^T X + K) = 0.$$

Note that K is confined to the term

$$(XBR^{-1} + K^T)R(R^{-1}B^T X + K) \succeq 0$$

$$\implies \quad K = -R^{-1}B^T X.$$

and Algebraic Riccati Equation (ARE) in X

$$A^T X + XA - XBR^{-1}B^T X + Q = 0.$$