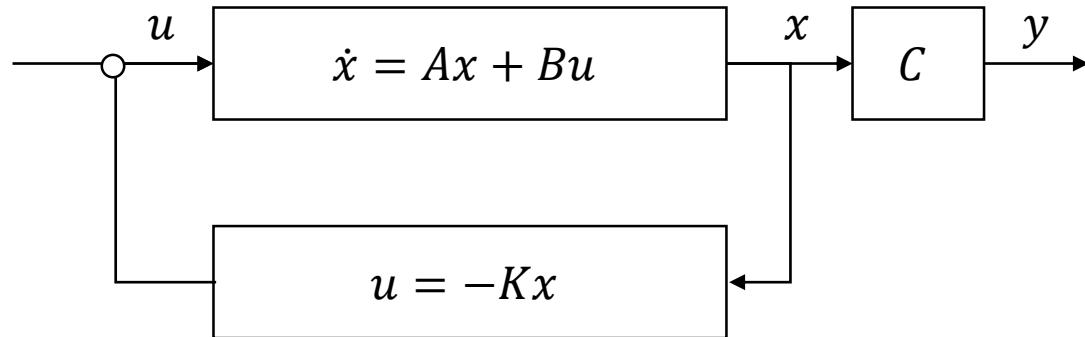


Modern Control Theory

state-space model analysis



State-space model: DC motor

Motor **speed** control

$$\mathbf{x} = [\dot{\theta} \quad i]^T$$

$$\mathbf{y} = \dot{\theta}$$

$$\mathbf{u} = E_a$$

- State equation

$$\dot{\mathbf{x}} = \begin{bmatrix} -\frac{b}{J} & \frac{K_t}{J} \\ -\frac{K_b}{L} & -\frac{R}{L} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} \mathbf{u}$$

- Output equation

$$\mathbf{y} = [1 \quad 0] \mathbf{x}$$

Motor **position** control

$$\mathbf{x} = [\dot{\theta} \quad \theta \quad i]^T$$

$$\mathbf{y} = \theta$$

$$\mathbf{u} = E_a$$

- State equation

$$\dot{\mathbf{x}} = \begin{bmatrix} -\frac{b}{J} & 0 & \frac{K_t}{J} \\ 1 & 0 & 0 \\ -\frac{K_b}{L} & 0 & -\frac{R}{L} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1/L \end{bmatrix} \mathbf{u}$$

- Output equation

$$\mathbf{y} = [0 \quad 1 \quad 0] \mathbf{x}$$

Transfer function to state-space model

Vehicle longitudinal motion

$$\mathbf{x} = [v \quad a]^T$$

$$\mathbf{y} = v$$

$$\mathbf{u} = a^d$$

- State equation

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & -1/\tau \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1/\tau \end{bmatrix} \mathbf{u}$$

- Output equation

$$\mathbf{y} = [1 \quad 0] \mathbf{x}$$

Vehicle lateral motion

$$\begin{bmatrix} \dot{e}_1 \\ \ddot{e}_1 \\ \dot{e}_2 \\ \ddot{e}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{2C_{\alpha_f}+2C_{\alpha_r}}{mv_x} & \frac{2C_{\alpha_f}+2C_{\alpha_r}}{m} & \frac{-2C_{\alpha_f}l_f+2C_{\alpha_r}l_r}{mv_x} \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{2C_{\alpha_f}l_f-2C_{\alpha_r}l_r}{I_z v_x} & \frac{2C_{\alpha_f}l_f-2C_{\alpha_r}l_r}{I_z} & -\frac{2C_{\alpha_f}l_f^2+2C_{\alpha_r}l_r^2}{I_z v_x} \end{bmatrix} \begin{bmatrix} e_1 \\ \dot{e}_1 \\ e_2 \\ \dot{e}_2 \end{bmatrix} +$$
$$+ \begin{bmatrix} 0 \\ \frac{2C_{\alpha_f}}{m} \\ 0 \\ \frac{2C_{\alpha_f}l_f}{I_z} \end{bmatrix} \delta + \begin{bmatrix} 0 \\ -\frac{2C_{\alpha_f}l_f-2C_{\alpha_r}l_r}{mv_x} - v_x \\ 0 \\ -\frac{2C_{\alpha_f}l_f^2+2C_{\alpha_r}l_r^2}{I_z v_x} \dot{\psi}_d \end{bmatrix} \psi_{des}$$

Linear and Nonlinear system

Linear systems

- Time-invariant linear system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$



$$T(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

- Time-varying linear system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$$y(t) = C(t)x(t) + D(t)u(t)$$



$$T(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

Nonlinear systems

- Time-invariant linear system

$$\dot{x}(t) = f(x(t), u(t))$$

$$y(t) = g(x(t), u(t))$$



$$T(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

- Time-varying linear system

$$\dot{x}(t) = f(x(t), u(t), t)$$

$$y(t) = g(x(t), u(t), t)$$



$$T(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

Linear and Nonlinear system

Linear systems

- Time-invariant linear system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$\dot{x} = -2 \cdot x + u$$

$$y(t) = Cx(t) + Du(t)$$

$$y = x$$

- Time-varying linear system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$$\dot{x} = -2t \cdot x + u$$

$$y(t) = C(t)x(t) + D(t)u(t)$$

$$y = x$$

Nonlinear systems

- Time-invariant linear system

$$\dot{x}(t) = f(x(t), u(t))$$

$$\dot{x} = -x^3 + u$$

$$y(t) = g(x(t), u(t))$$

$$y = x$$

- Time-varying linear system

$$\dot{x}(t) = f(x(t), u(t), t)$$

$$\dot{x} = -x^3 e^{-t} + u$$

$$y(t) = g(x(t), u(t), t)$$

$$y = x$$

State-space model analysis

Model analysis

$$\mathbf{x} = [\dot{\theta} \quad i]^T$$

$$\mathbf{y} = \dot{\theta}$$

$$\mathbf{u} = E_a$$

- State equation

$$\dot{\mathbf{x}} = \begin{bmatrix} -\frac{b}{J} & \frac{K_t}{J} \\ -\frac{K_b}{L} & -\frac{R}{L} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} \mathbf{u}$$

- Output equation

$$\mathbf{y} = [1 \quad 0] \mathbf{x}$$

Transfer function

- Stability, percent overshoot, rise time, settling time...

State-space model

- Stability, controllability, observability...

Eigenvector & eigenvalue

Eigenvalue

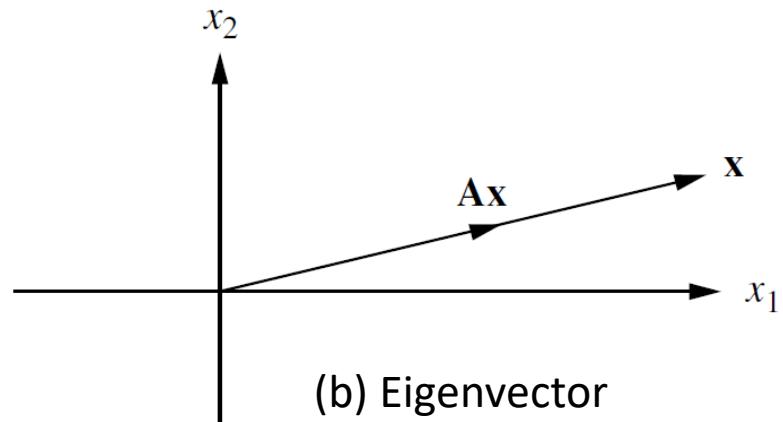
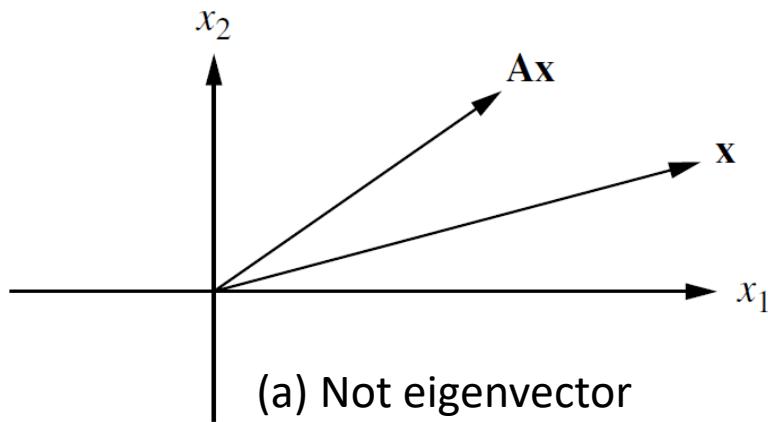
$$\mathbf{A}\mathbf{x}_i = \lambda_i \mathbf{x}_i$$

Eigenvector

$$\mathbf{x}_i = (\lambda_i \mathbf{I} - \mathbf{A})^{-1} \mathbf{0} = \frac{\text{adj}(\lambda_i \mathbf{I} - \mathbf{A})}{\det(\lambda_i \mathbf{I} - \mathbf{A})} \mathbf{0}$$

For nonzero solution \mathbf{x}

$$\det(\lambda_i \mathbf{I} - \mathbf{A}) = \mathbf{0}$$



Eigenvector & eigenvalue

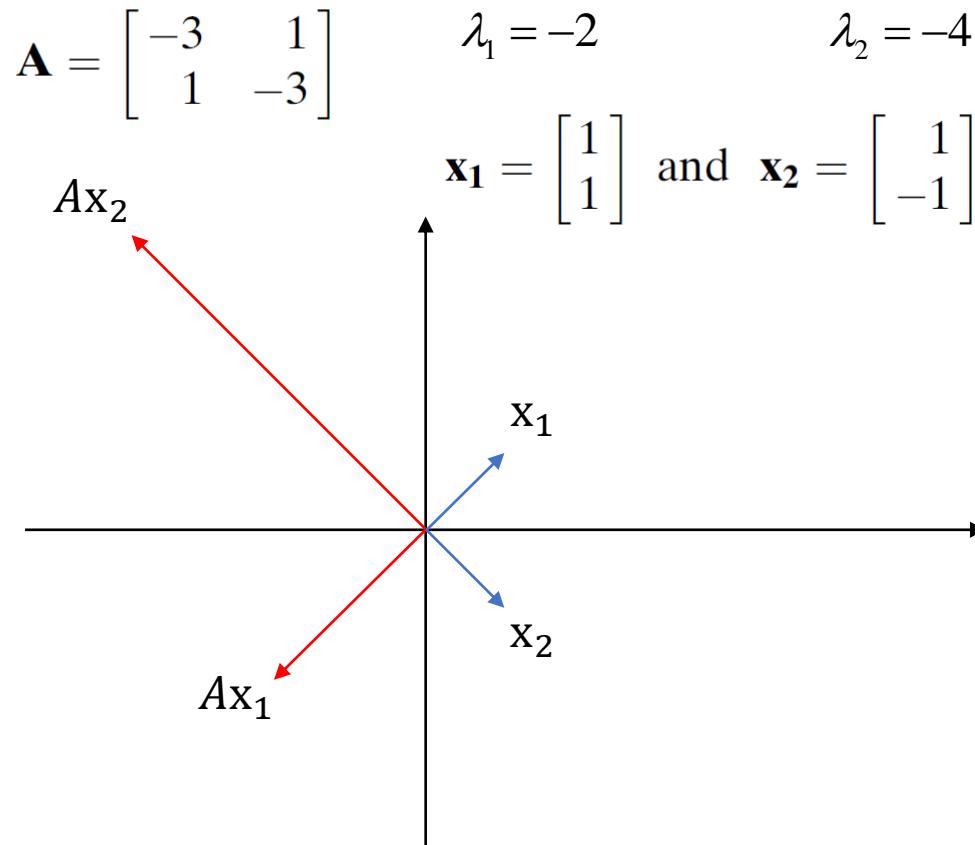
$$\mathbf{A} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \quad \det(\lambda \mathbf{I} - \mathbf{A}) = \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \right|$$
$$= \begin{vmatrix} \lambda + 3 & -1 \\ -1 & \lambda + 3 \end{vmatrix}$$
$$= \lambda^2 + 6\lambda + 8 \quad \longrightarrow \lambda = -2, \text{ and } -4$$

$$\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$\lambda_1 = -2 \qquad \qquad \qquad x_1 = x_2 \qquad \qquad \qquad \mathbf{x} = \begin{bmatrix} c \\ c \end{bmatrix}$$
$$\begin{aligned} -3x_1 + x_2 &= -2x_1 \\ x_1 - 3x_2 &= -2x_2 \end{aligned}$$

$$\lambda_2 = -4 \qquad \qquad \mathbf{x} = \begin{bmatrix} c \\ -c \end{bmatrix}$$

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Eigenvector & eigenvalue



Why eigenvalues are important?

$$\det(\lambda\mathbf{I} - \mathbf{A}) = 0$$

State equation solution: traditional

Given state equation,

$$\dot{x}(t) = ax(t) + bu(t) \quad x(t_0) = x_0$$

consider scalar state



$$e^{-a(t-t_0)}x(t)$$

$$\begin{aligned} \frac{d}{dt}(e^{-a(t-t_0)}x(t)) &= e^{-a(t-t_0)}\dot{x}(t) - e^{-a(t-t_0)}ax(t) \\ &= e^{-a(t-t_0)}bu(t) \end{aligned}$$

integrate from t_0 to t



$$\begin{aligned} e^{-a(t-t_0)}x(t) - e^{-a(t-t_0)}x(t_0) &= \int_{t_0}^t \frac{d}{dt}(e^{-a(\tau-t_0)}x(\tau)) d\tau \\ &= \int_{t_0}^t e^{-a(\tau-t_0)}bu(\tau) d\tau. \end{aligned}$$

multiplying through by $e^{a(t-t_0)}$

$$x(t) = e^{a(t-t_0)}x_0 + \int_{t_0}^t e^{a(t-\tau)}bu(\tau) d\tau$$

State equation solution: Laplace transform

Given state equation,

$$\dot{x}(t) = ax(t) + bu(t) \quad x(t_0) = x_0$$



← Laplace transform

$$sX(s) - x_0 = aX(s) + bU(s)$$



$$X(s) = \frac{1}{s-a}x_0 + \frac{b}{s-a}U(s)$$



← Laplace transform

$$x(t) = e^{at}x_0 + e^{at} * bu(t)$$

$$= e^{at}x_0 + \int_0^t e^{a(t-\tau)}bu(\tau) d\tau$$

State equation solution

$$\dot{x}(t) = ax(t) + bu(t) \quad x(t_0) = x_0 \quad \longrightarrow \quad \dot{x}(t) = Ax(t) + Bu(t) \quad x(t_0) = x_0$$

$$x(t) = e^{at}x_0 + \int_0^t e^{a(t-\tau)}bu(\tau) d\tau \quad \longrightarrow$$

Matrix exponential

$$e^{at} = 1 + at + \frac{1}{2}a^2t^2 + \frac{1}{6}a^3t^3 + \dots$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} a^k t^k$$

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3 + \dots$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k$$

State equation solution

$$\dot{X}(t) = AX(t) \quad X(t_0) = I_{n \times n}$$

assume an infinite power series form for the solution

$$\begin{aligned} X(t) &= \sum_{k=0}^{\infty} X_k (t - t_0)^k \\ &\quad \downarrow \\ \sum_{k=0}^{\infty} (k+1) X_{k+1} (t - t_0)^k &= A \left(\sum_{k=0}^{\infty} X_k (t - t_0)^k \right) \\ &= \sum_{k=0}^{\infty} A X_k (t - t_0)^k \end{aligned}$$

$$X_{k+1} = \frac{1}{k+1} A X_k \quad k \geq 0 \xrightarrow{X_0 = I} X_k = \frac{1}{k!} A^k \quad k \geq 0 \xrightarrow{\text{Substituting}} X(t) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k (t - t_0)^k$$

Matrix exponential properties

$$X(t) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k (t - t_0)^k$$

1. e^{At} is the unique matrix satisfying

$$\frac{d}{dt} e^{At} = A e^{At} \quad e^{At} \Big|_{t=0} = I_n$$

2. For any t_1 and t_2 , $e^{A(t_1+t_2)} = e^{At_1} e^{At_2}$. As a direct consequence, for any t

$$I = e^{A(0)} = e^{A(t-t)} = e^{At} e^{-At}$$

Thus e^{At} is invertible (nonsingular) for all t with inverse

$$[e^{At}]^{-1} = e^{-At}$$

3. A and e^{At} commute with respect to matrix multiplication, that is, $A e^{At} = e^{At} A$ for all t .

4. $[e^{At}]^T = e^{A^T t}$ for all t .

5. For any real $n \times n$ matrix B , $e^{(A+B)t} = e^{At} e^{Bt}$ for all t if and only if $AB = BA$, that is, A and B commute with respect to matrix multiplication.

□

$$e^{At} \neq [e^{a_{ij}t}]$$

Controllability

$$x(t) = e^{at}x_0 + \int_0^t e^{a(t-\tau)}bu(\tau) d\tau$$

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau) d\tau$$

Example

$$X(t) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k (t - t_0)^k \quad e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3 + \dots$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad A^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad A^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad A^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{1}{2}t^2 & 0 \\ 0 & 0 & 0 & \frac{1}{2}t^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \frac{1}{6}t^3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & t & \frac{1}{2}t^2 & \frac{1}{6}t^3 \\ 0 & 1 & t & \frac{1}{2}t^2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \Rightarrow e^{At} = \begin{bmatrix} 1 & t & \frac{1}{2}t^2 & \cdots & \frac{1}{(n-1)!}t^{n-1} \\ 0 & 1 & t & \cdots & \frac{1}{(n-2)!}t^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & t \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Example

$$X(t) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k (t - t_0)^k$$

$$\begin{aligned} e^{At} &= I + At + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3 + \dots \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k \end{aligned}$$

$$A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{bmatrix} \quad A^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2^k & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_{n-1}^k & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n^k \end{bmatrix}$$

$$e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2^k & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_{n-1}^k & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n^k \end{bmatrix} t^k = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{1}{k!} \lambda_1^k t^k & 0 & \cdots & 0 & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{1}{k!} \lambda_2^k t^k & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sum_{k=0}^{\infty} \frac{1}{k!} \lambda_{n-1}^k t^k & 0 \\ 0 & 0 & \cdots & 0 & \sum_{k=0}^{\infty} \frac{1}{k!} \lambda_n^k t^k \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_{n-1} t} & 0 \\ 0 & 0 & \cdots & 0 & e^{\lambda_n t} \end{bmatrix}$$

Properties

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau) d\tau$$
$$e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k$$

$$\frac{d}{dt} e^{At} = \frac{d}{dt} \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k = \sum_{k=0}^{\infty} \frac{1}{k!} k A^k t^{k-1} = A \sum_{i=0}^{\infty} \frac{1}{i!} A^i t^i = Ae^{At}$$

→ $\frac{d}{dt} e^{At} = Ae^{At}$

→ $s\mathcal{L}\{e^{At}\} - e^0 = A\mathcal{L}\{e^{At}\} \rightarrow (sI - A)\mathcal{L}\{e^{At}\} = I$

→ $\mathcal{L}\{e^{At}\} = (sI - A)^{-1}$

→ $e^{At} = \mathcal{L}\{(sI - A)^{-1}\}$

Properties

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau) d\tau$$

a change of variables $\lambda = t - \tau$ and took $u(t) = 1$

$$\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = \int_0^t e^{A\lambda}d\lambda B$$

$$\begin{aligned}\longrightarrow \quad & \frac{d}{dt}e^{At} = Ae^{At} \\ \longrightarrow \quad & \int_0^t \frac{d}{d\tau}e^{A\tau} d\tau = \int_0^t Ae^{A\tau} d\tau = e^{At} - I \\ \longrightarrow \quad & \int_0^t e^{A\tau} d\tau = A^{-1}(e^{At} - I)\end{aligned}$$

$$\boxed{\int_0^t e^{A\tau} d\tau B = A^{-1}(e^{At} - I)B}$$

Example

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau) d\tau$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad x(0) = (1, 0)^T$$

$$y = x.$$

1) $e^{At} = \mathcal{L}\{(sI - A)^{-1}\}$

2) $\int_0^t e^{A\tau} d\tau B = A^{-1}(e^{At} - I)B$

First we find $\exp(At)$ using (say) Laplace transform

$$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{2} \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix}$$



$$\exp(At) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-2t} - e^{-t} \end{bmatrix}$$

2) $\int_0^t e^{A(t-\tau)} Bu(\tau) d\tau = A^{-1}(\exp(At) - I)B = \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{e^{-2t}}{2} \\ e^{-t} - e^{-2t} \end{bmatrix}.$

$$y(t) = x(t) = \underbrace{\begin{bmatrix} 2e^{-t} - e^{-2t} \\ 2e^{-2t} - 2e^{-t} \end{bmatrix}}_{\text{Zero-input response}} + \underbrace{\begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}}_{\text{Zero-state response}} = \begin{bmatrix} e^{-t} + \frac{1}{2} - \frac{1}{2}e^{-2t} \\ e^{-2t} - e^{-t} \end{bmatrix}.$$