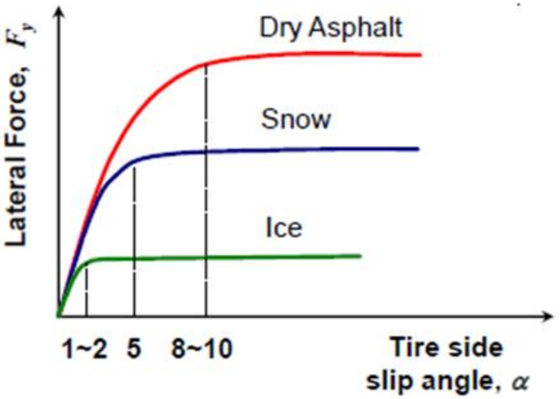
Bicycle Model of Lateral Vehicle Dynamics

- The slope at low slip values is called the cornering stiffness C_α

$$F_y = C_\alpha \alpha, \quad |\alpha| < 5 \text{ deg}$$

- However, the tire model will be *no longer linear at large slip angles !*

➔ *Lateral tire force* $= f(F_z, \mu, \alpha, \cdots)$



Example 1 (tire_cornering)

linear

$$F_y = C_\alpha \alpha$$

nonlinear

$$F_y = C_\alpha \frac{\mu}{K} tan^{-1}\left(\frac{K}{\mu}\alpha\right)$$

Calpha = 34500
mu = 0.85
K = 18
alpha = 1~10

Bicycle Model of Lateral Vehicle Dynamics

$$\dot{v}_x = 0$$

$$\dot{v}_y = \frac{1}{m}\left(F_{yf} + F_{yr}\right) - v_y\dot{\psi}$$

$$\dot{\gamma} = \ddot{\psi} = \frac{1}{I_{zz}}\left(l_{yf}F_{yf} - l_{yr}F_{yr}\right)$$

$$\dot{X} = v_x\cos(\psi) - v_y\sin(\psi)$$

$$\dot{Y} = v_x\sin(\psi) + v_y\cos(\psi)$$

$$\dot{\psi} = \gamma$$

m = 1765
Iz = 4828
lf = 1.4
lr = 1.7
Cf = 39500
Cr = 38500

Step steer for J-turn

Example 2 (vehicle_motion)
(vehicle_motion_nonlinear)

Bicycle Model of Lateral Vehicle Dynamics

Example 3 (global_motion)

$$\dot{X} = V\cos(\psi + \beta)$$
$$\dot{Y} = V\sin(\psi + \beta)$$

무시할 수 있다고 가정

Example 4 (motion_sim) **para_set.m** **motion_plot.m**

Steer angle의 step input에 따른 선형/비선형 차량 모델의 움직임 확인

Example 5 (motion_sim_lonlat)

Steer angle의 step input에 따른 선형/비선형 차량 모델의 움직임 확인
차량의 종방향 속도 제어(PI)를 포함한 움직임

State feedback Control

- Vehicle parameters for simulation
  - 1986 Pontiac 6000 STE Sedan

$$m = 1573kg, I_z = 2873kgm^2$$

$$l_f = 1.1m, l_r = 1.58m$$

$$C_{\alpha f} = C_{\alpha r} = 8e4N/rad$$

  - Assume $V_x = 30m/sec$

- See **lat_dyn.m**

- Eigenvalues of matrix A
  - $\text{eig}(A) = [0 \; -6.8308 + 5.0278i \; -6.8308 - 5.0278i \; 0]^T$
  - The open loop matrix A has two eigenvalues at the origin and is unstable
  - The system has to be stabilized by feedback

State feedback Control

lat_dyn.m

```
function [A,B,C,D] =lateral_model(V)
% This function calculates the matrices for the state space model of
% the lateral vehicle system, for a given longitudinal vehicle speed V
% function [A,B,C,D] = lateral_model(V)
% V - longitudinal velocity
% B - first column steering input, second column desired psi_dot
% C - all 4 states as outputs
% D - zeros (4 x 2)
% Vehicle Parameters (1986 Pontiac 6000 STE Sedan)

m = 1573;
Iz = 2873;
lf = 1.1;
lr = 1.58;
Cf = 80000;
Cr = 80000;
Vx = V;
I_psi = Iz;

a22 = -(2*Cf+2*Cr)/(m*Vx);
a23 = 2*(Cf+Cr)/m;
a24 = - (2*Cf*lf - 2*Cr*lr)/(m*Vx);
a42 = -(2*lf*Cf-2*lr*Cr)/(I_psi*Vx);
a43 = 2*(lf*Cf-lr*Cr)/I_psi;
a44 = -(2*lf*lf*Cf+2*lr*lr*Cr)/(I_psi*Vx);
b21 = 2*Cf/m;
b41 = 2*lf*Cf/I_psi;
b22 = -Vx -2*(lf*Cf-lr*Cr)/(m*Vx);
b42 = -2*(lf*lf*Cf+lr*lr*Cr)/(I_psi*Vx);

A_lat = [0 1 0 0; 0 a22 a23 a24;
         0 0 0 1; 0 a42 a43 a44;];
B1_lat = [0 b21 0 b41]'; %Steering angle input
B2_lat = [0 b22 0 b42]'; %psi_des_dot input
C_lat = eye(4);
D1_lat = zeros(4,1);
D2_lat = zeros(4,1);

A = A_lat;
B = [B1_lat B2_lat];
C = C_lat;
D = [D1_lat D2_lat];
```

State feedback Control

- Controllability

  - The system is controllable if, using appropriate control inputs, the states can be moved in any direction in the state space

  - The pair (A, B) is controllable if and only if the rank of the controllability matrix, $\mathbf{C}$, is n (n is the system order, i.e., dimensions of A). The controllability matrix, $\mathbf{C}$, is given by

$$\mathbf{C} = [B \ AB \ A^2B \ \cdots \ A^{n-1}B]$$

  - $\text{rank}(A, B) = 4$ → Controllabe!!

```
>> B1=B(:,1)

B1 =

        0
   101.7165
        0
    61.2600

>> %The command ctrb, uses A and B as inputs and
returns the controllability matrix
>> C=ctrb(A,B1)

C =

  1.0e+005 *

        0    0.0010   -0.0059    0.1593
   0.0010   -0.0059    0.1593   -1.7511
        0    0.0006   -0.0033    0.0011
   0.0006   -0.0033    0.0011    0.2226

>> rank(C)

ans =

     4
```

State feedback Control

- State feedback control law

$$\delta = -Kx = -k_1 e_1 - k_2 \dot{e}_2 - k_3 e_3 - k_4 \dot{e}_2$$

- Pole Placement
  - The eigenvalues of the closed-loop matrix $A - B_1 K$ can be placed at any desired locations

- Closed-loop system using state feedback controller

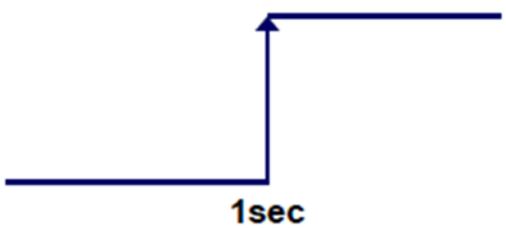
$$\dot{x} = (A - B_1 K)x + B_2 \dot{\psi}_{des}$$

- Matlab Command
  - K=place(A,B1,P)
  - This command yields a feedback matrix K such that the eigenvalues of the matrix $A - B_1 K$ are at the desired locations specified in the vector P

State feedback Control

- Longitudinal speed of vehicle, $V_x = 30m/sec$
- Road profile
  - Initially straight and then becomes circular with a radius of 1000m starting at a time of 1 sec
- Desired yaw rate

$$\dot{\psi}_{des} = \frac{V_x}{R} = 0.03rad/sec = 1.72deg/sec$$



$$\dot{\psi}_{des} = 0.03rad/sec$$

- Eigenvalues of $A - B_1 K$

$$P = [-5-3j \;\; -5+3j \;\; -7 \; 10]^T$$

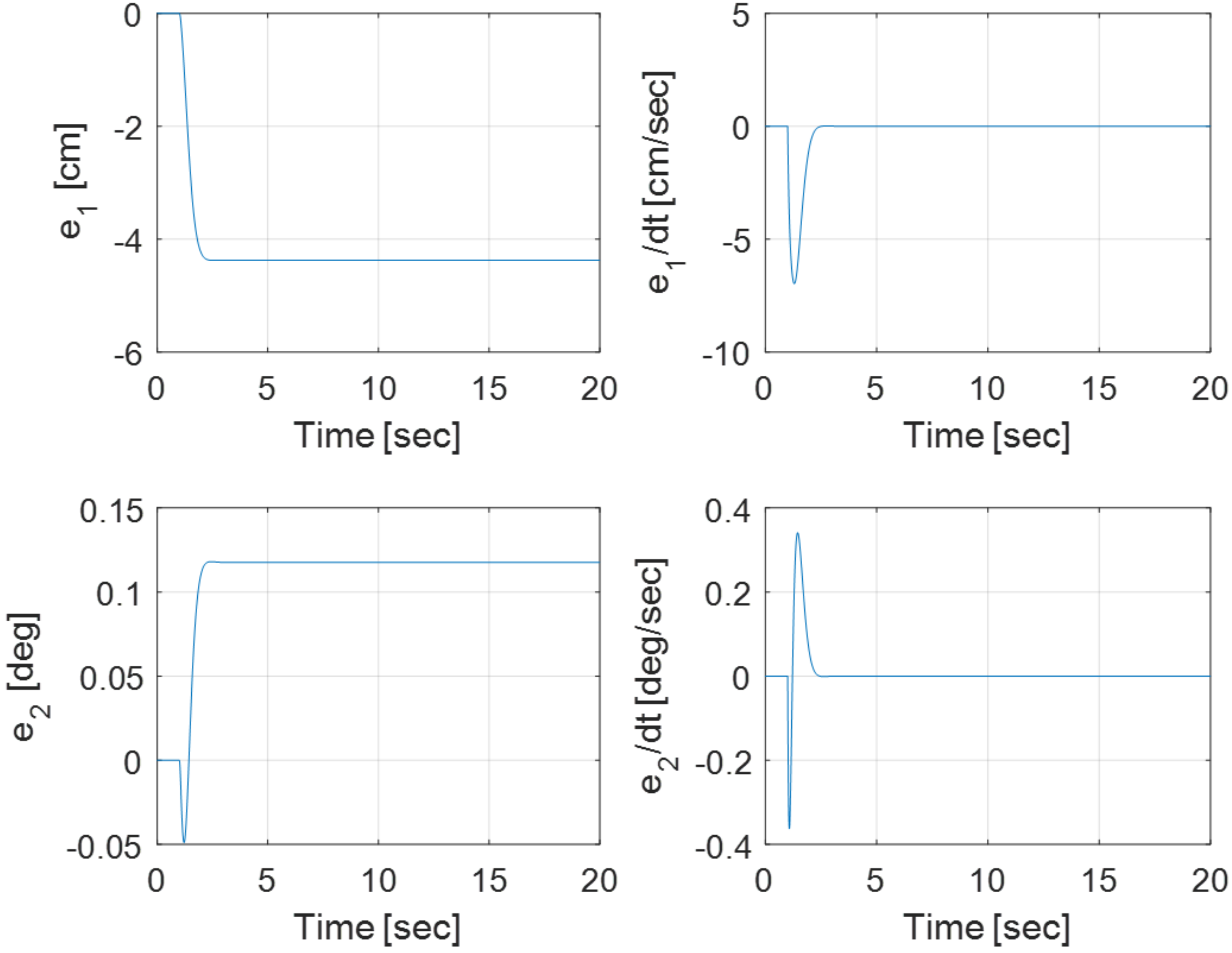
State feedback Control

sfcontrol.m

```
% state_feedback_controller.m
clear all

Vx=30; % Longitudinal vehicle speed (m/s)
% Determination of feedback gain via pole placement
[A,B,C,D] = lateral_model(Vx);

P = [-5-3*j -5+3*j -7 -10]; % Eignevalues of A-B1K
K = place(A,B(:,1),P);

% Closed-loop system with the state feedback controller
Ac = A-B(:,1)* K;
Bc = B(:,2);
Cc = eye(4);
Dc = 0;
sys = ss(Ac,Bc,Cc,Dc);

t = 0:0.01:20;
R = 1000;

% Define Desired yaw rate as a step input
T = floor(length(t)/20);
u = [zeros(1,T) (Vx/R)*ones(1,length(t)-T)]’;

[y,t,x] = lsim(sys,u,t);
```

```
figure
subplot(2,2,1)
plot(t,x(:,1)*100)
grid on
xlabel('Time [sec]')
ylabel('e_1 [cm]')
xlim([0 20])
subplot(2,2,2)
plot(t,x(:,2)*100)
grid on
xlabel('Time [sec]')
ylabel('e_1/dt [cm/sec]')
xlim([0 20])
subplot(2,2,3)
plot(t,x(:,3)*180/pi)
grid on
xlabel('Time [sec]')
ylabel('e_2 [deg]')
xlim([0 20])
subplot(2,2,4)
plot(t,x(:,4)*180/pi)
grid on
xlabel('Time [sec]')
ylabel('e_2/dt [deg/sec]')
xlim([0 20])
```

State feedback Control



Steady State Error from Dynamic Equations

➢ Model of the closed-loop lateral system under state feedback

$$\dot{x} = (A - B_1 K)x + B_2 \dot{\psi}_{des}$$

- Due to the presence of $B_2 \dot{\psi}_{des}$ term, the tracking errors will not all converge to zero when the vehicle is traveling on a curve, even though the matrix $A - B_1 K$ is asymptotically stable

➢ Investigate whether the addition of a feedforward term to state feedback can ensue zero steady state errors on a curve

$$\delta = -Kx + \delta_{ff}$$

- δ_{ff} : feedforward term that attempts to compensate for the road curvature
- Closed-loop system

$$\dot{x} = (A - B_1 K)x + B_1 \delta_{ff} + B_2 \dot{\psi}_{des}$$

Steady State Error from Dynamic Equations

- Laplace transform with zero initial conditions

$$X(s) = \left[sI - (A - B_1K)\right]^{-1}\left\{B_1\mathcal{L}(\delta_{ff}) + B_2\mathcal{L}(\dot{\psi}_{des})\right\}$$

  - If the vehicle travels at constant speed V_x on a road with a constant radius R , then

$$\dot{\psi}_{des} = \text{constant} = \frac{V_x}{R} \Rightarrow \mathcal{L}(\dot{\psi}_{des}) = \frac{V_x}{Rs}$$

  - If the feedforward term is constant, then

$$\delta_{ff} = \text{constant} \Rightarrow \mathcal{L}(\delta_{ff}) = \frac{\delta_{ff}}{s}$$

- Steady state tracking error (Final Value Theorem)

$$x_{ss} = \lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) = -(A - B_1K)^{-1}\left\{b_1\delta_{ff} + B_2\frac{V_x}{R}\right\}$$

Steady State Error from Dynamic Equations

➢ Steady state error

$$x_{ss} = \begin{bmatrix} \frac{\delta_{ff}}{k_1} \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{k_1}\frac{mV_x^2}{R(l_f+l_r)}\left(\frac{l_r}{2C_{\alpha f}} - \frac{l_f}{2C_{\alpha r}} + \frac{l_f}{2C_{\alpha r}}k_3\right) - \frac{1}{k_1 R}\left(l_f + l_r - l_r k_3\right) \\ 0 \\ \frac{1}{2RC_{\alpha r}(l_f+l_r)}\left(-2C_{\alpha r}l_f l_r - 2C_{\alpha r}l_r^2 + l_f m V_x^2\right) \\ 0 \end{bmatrix}$$

- Lateral position error e_1 can be made zero by appropriate choice of δ_{ff}
- However, δ_{ff} cannot influence on the steady state yaw error

➢ Yaw angle error

$$e_{2ss} = \frac{1}{2RC_{\alpha r}(l_f+l_r)}\left(-2C_{\alpha r}l_f l_r - 2C_{\alpha r}l_r^2 + l_f m V_x^2\right)$$
$$= -\frac{l_r}{R} + \frac{l_f}{2C_{\alpha r}(l_f+l_r)}\frac{mV_x^2}{R}$$

The yaw angle error has a steady state term that cannot be corrected, no matter how the feedforward steering angle is chosen

Steady State Error from Dynamic Equations

- Steady state lateral position error can be made zero if the feedforward steering angle is chosen as:

$$\delta_{ff} = \frac{mV_x^2}{RL}\left(\frac{l_r}{2C_{\alpha f}} - \frac{l_f}{2C_{\alpha r}} + \frac{l_f}{2C_{\alpha r}}k_3\right) + \frac{L}{R} - \frac{l_r}{R}K_3$$

$$= \frac{L}{R} + K_v a_y - k_3 \underbrace{\left(\frac{l_r}{R} - \frac{l_f}{2C_{\alpha r}}\frac{mV_x^2}{RL}\right)}_{= -e_{2ss}}$$

$$L = l_f + l_r,\ a_y = \frac{V_x^2}{R}$$

where

$$K_v = \frac{l_r m}{2C_{\alpha f}(l_f + l_r)} - \frac{l_f m}{2C_{\alpha r}(l_f + l_r)} = \frac{m_f}{2C_{\alpha f}} - \frac{m_r}{2C_{\alpha r}}$$

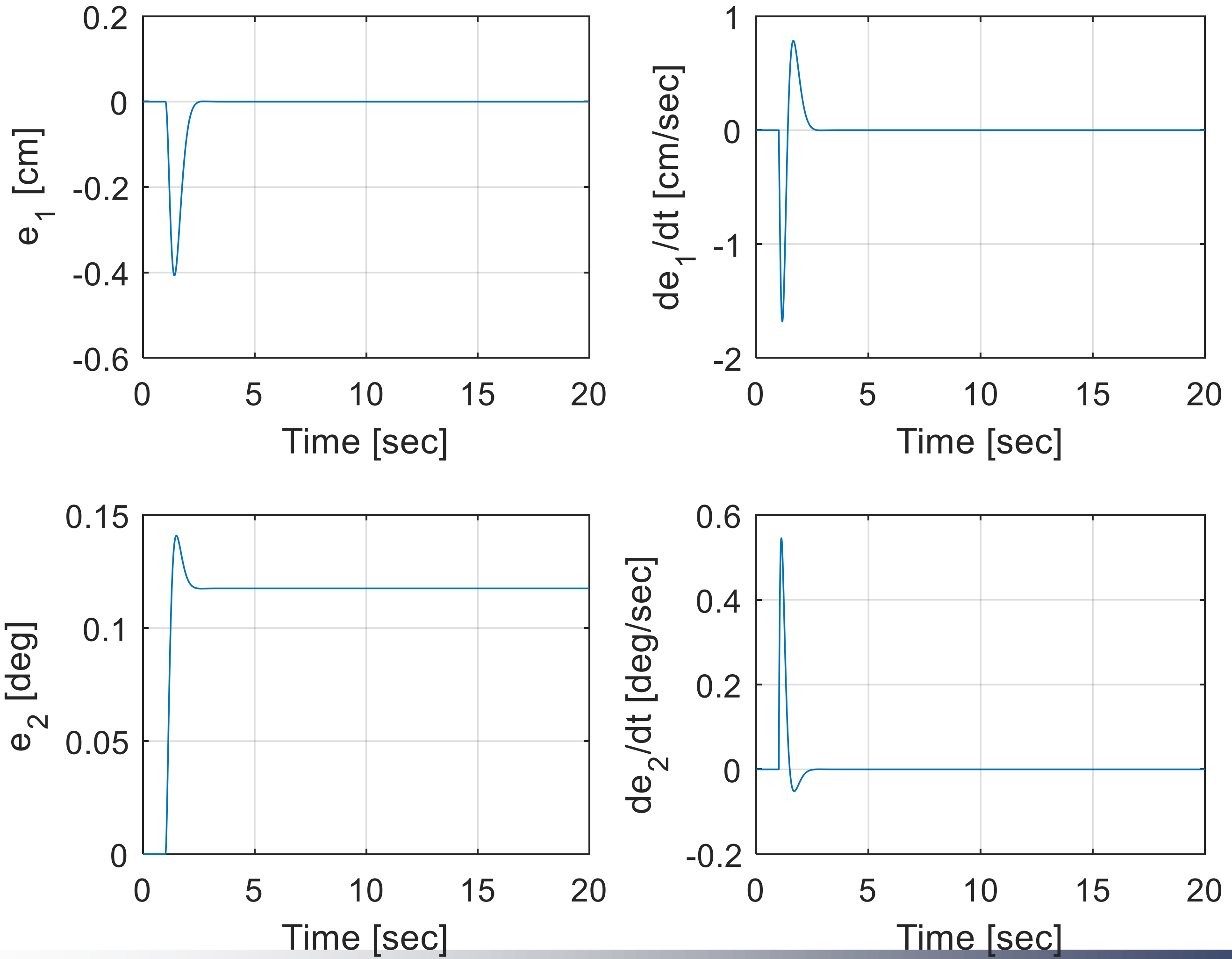
$$m_f = m\frac{l_r}{L},\ m_r = m\frac{l_f}{L}$$

Steady State Error from Dynamic Equations

ffcontrol.m

```
% Effect of Feedforward control on Lane Keeping
clear all
% Vehicle Parameters (1986 Pontiac 6000 STE Sedan)
m = 1573; Iz = 2873;
lf = 1.1; lr = 1.58;
Cf = 80000; Cr = 80000;
L = (lf+lr);
R = 1000;
Vx=30;
% Determination of feedback gain via pole placement
[A,B,C,D] = lateral_model(Vx);
% Eignevalues of A-B1K
P = [-5-3*j -5+3*j -7 -10];
% Feedback matrix K such that the eigenvalues of the matrix
% A-B1K are at the desired locations specified in the vector P
K = place(A,B(:,1),P);
% Calculation of feedforward steering angle
ay = Vx^2/R;
Kv = lr*m/(2*Cf*L)-lf*m/(2*Cr*L);
delta_ff = L/R+Kv*ay-K(3)*(lr/R-lf/(2*Cr)*m*ay/L);
% Steady State Errors
e1ss = (-(1/K(1))*(m*Vx^2)/(R*L)*(lr/(2*Cf)-lf/(2*Cr)+lf/(2*Cr)*K(3))-(1/(K(1)*R))*(L-lr*K(3))...
        +delta_ff/K(1))*100
e2ss = (1/(2*R*Cr*L)*(-2*Cr*lf*lr-2*Cr*lr^2+lf*m*Vx^2))*180/pi
Ac = A-B(:,1)* K;
Bc = B;
Cc = eye(4);
Dc = 0;
sys = ss(Ac,Bc,Cc,Dc);
t = 0:0.01:20;
% Define Input vector, u
T = floor(length(t)/20);
u = [zeros(1,T) delta_ff*ones(1,length(t)-T);
zeros(1,T) (Vx/R)*ones(1,length(t)-T)]';
[y,t,x] = lsim(sys,u,t);
figure
subplot(2,2,1)
plot(t,x(:,1)*100)
grid on
xlabel('Time [sec]')
ylabel('e_1 [cm]')
xlim([0 20])
subplot(2,2,2)
plot(t,x(:,2)*100)
grid on
xlabel('Time [sec]')
ylabel('de_1/dt [cm/sec]')
xlim([0 20])
subplot(2,2,3)
plot(t,x(:,3)*180/pi)
grid on
xlabel('Time [sec]')
ylabel('e_2 [deg]')
xlim([0 20])
subplot(2,2,4)
plot(t,x(:,4)*180/pi)
grid on
xlabel('Time [sec]')
ylabel('de_2/dt [deg/sec]')
xlim([0 20])
```

Steady State Error from Dynamic Equations



Steady State Error from Dynamic Equations

- Feedforward component of steering angle

$$\delta_{ff} = \frac{L}{R} + K_v a_y + k_3 e_{2ss}$$

- Steady state steering angle for zero lateral position error

$$\delta_{ss} = -Kx_{ss} + \delta_{ff} = -k_3 e_{2ss} + \delta_{ff} = \frac{L}{R} + K_v a_y$$

- Conclusions
  - The lateral position error e_1 can be made zero at steady state by appropriate choice of the feedforward input δ_{ff}
  - However, the steady state yaw error will be equal to

$$e_{2ss} = -\frac{l_r}{R} + \frac{l_f}{2C_{\alpha r}(l_f + l_r)} \frac{mV_x^2}{R}$$

and cannot be changed by the feedforward steering input