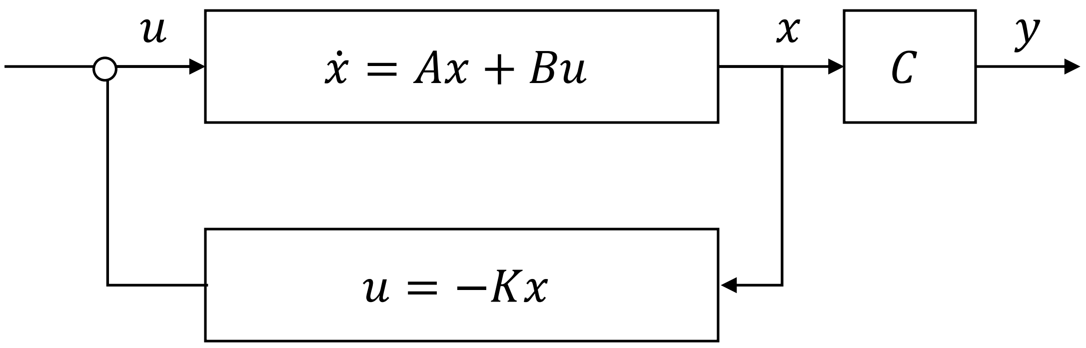
Modern Control Theory

Application: autonomous vehicle



Bicycle Model of Lateral Vehicle Dynamics

- At higher vehicle speeds
  - The assumption that the velocity at each wheel is in the direction of the wheel can be no longer made

➔ **Dynamic model instead of kinematic model must be developed**

- Assumptions
  - Road is level and flat
  - Aerodynamic forces are ignored
  - Vehicle body is rigid and suspension is not considered
  - The steering system is assumed to be non-flexible: the input is assumed to be a front wheel steering angle, but it could equally be a handwheel angle

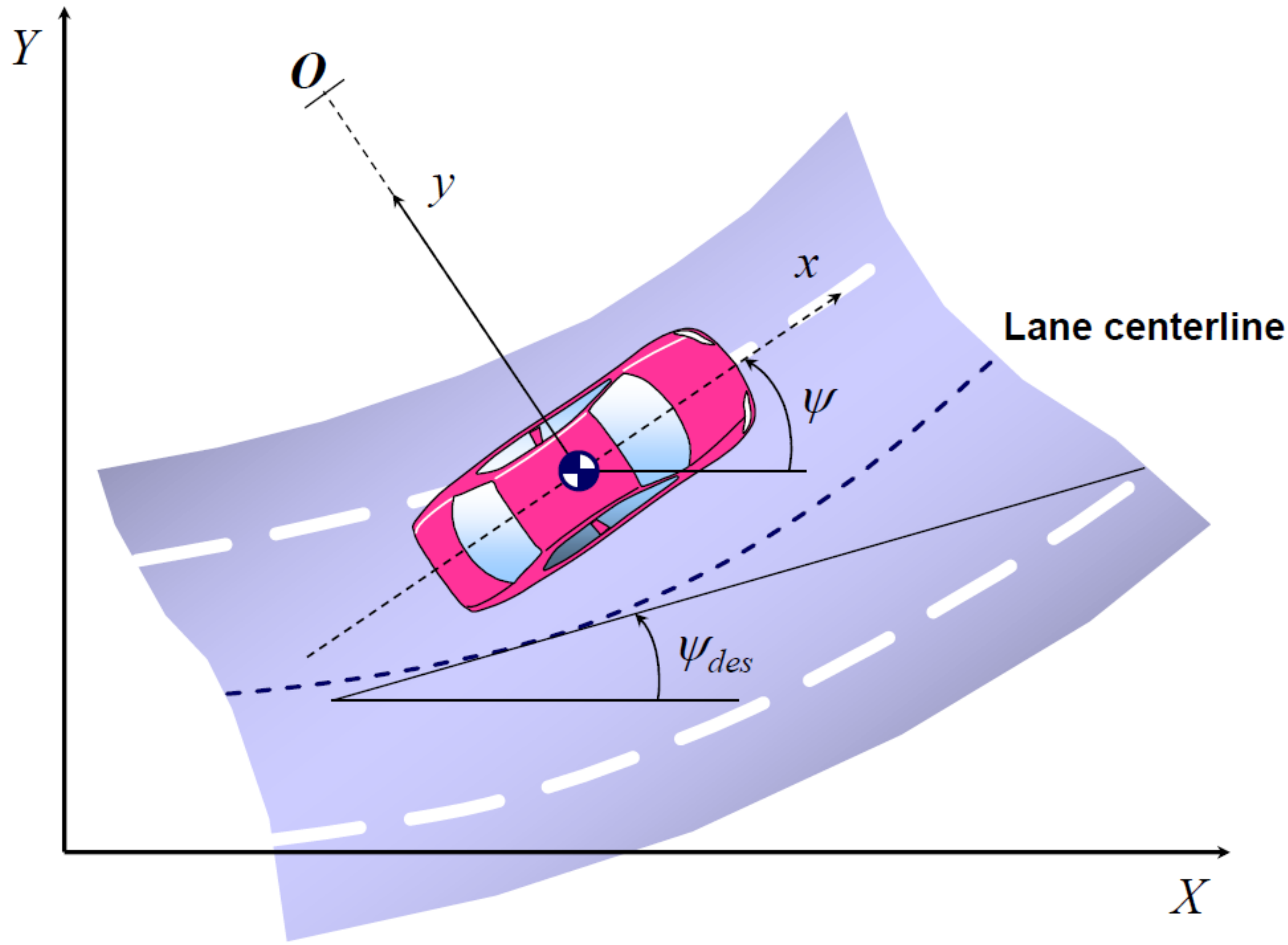
    (= front wheel angle steering ratio)
  - In general, 3 DOFs are actually needed to define the motion of car on a horizontal surface: forward velocity, lateral velocity, and yaw rate
  - However, forward velocity will be assumed constant and it becomes, therefore, a parameter rather than a system variable

Bicycle Model of Lateral Vehicle Dynamics

- Important omissions
  - Roll motion of sprung mass
  - Suspension effect, e.g. camber
  - Tire load transfer
  - Steering system compliance
  - …

- Bicycle model of the vehicle with 2DOF
  - Simplest representation of vehicle handling which still captures the key elements of behavior
  - The 2 DOF are represented by the
    - Vehicle lateral position, $y \rightarrow$ measured along the lateral axis of the vehicle to the point O which is the center of rotation of the vehicle
    - Vehicle yaw angle, $\psi \rightarrow$ measured w.r.t. the global X axis
  - The longitudinal velocity of the vehicle at CG is denoted by V_x

Bicycle Model of Lateral Vehicle Dynamics



Bicycle Model of Lateral Vehicle Dynamics

- Ignoring road bank angle and applying Newton's 2nd law for motion along the y axis

$$ma_y = F_{yf} + F_{yr} \quad \text{where} \quad a_y = \left(\frac{d^2 y}{dt^2}\right)_{inertial}$$

  - a_y is the inertial acceleration of the vehicle at the CG in the direction of y axis
  - F_{yf} and F_{yr} are the lateral tire forces of the front and rear wheels respectively

- Two terms contribute to a_y :

$$a_y = \ddot{y} + V_x \dot{\psi}$$

where

$\ddot{y}$: Centripetal acceleration

$V_x \dot{\psi}$: Due to motion along the y axis

- Equation for the lateral translational motion of the vehicle

$$m(\ddot{y} + V_x \dot{\psi}) = F_{yf} + F_{yr} \Longrightarrow mV_x(\dot{\beta} + \dot{\psi}) = F_{yf} + F_{yr}$$

$$\beta = \frac{V_y}{V_x} \Rightarrow \ddot{y} = \dot{y} \simeq V_x \dot{\beta}$$

Bicycle Model of Lateral Vehicle Dynamics

- Moment balance about the z axis

$$I_z\ddot{\psi} = l_f F_{yf} - l_r F_{yr}$$

- Tire Slip Angle
  - The lateral tire force is proportional to the slip angle (for small slip angles)
  - The slip angle of a tire is defined as the angle between the orientation of the tire and the orientation of the velocity vector of the wheel
  - Slip angle of the front wheel

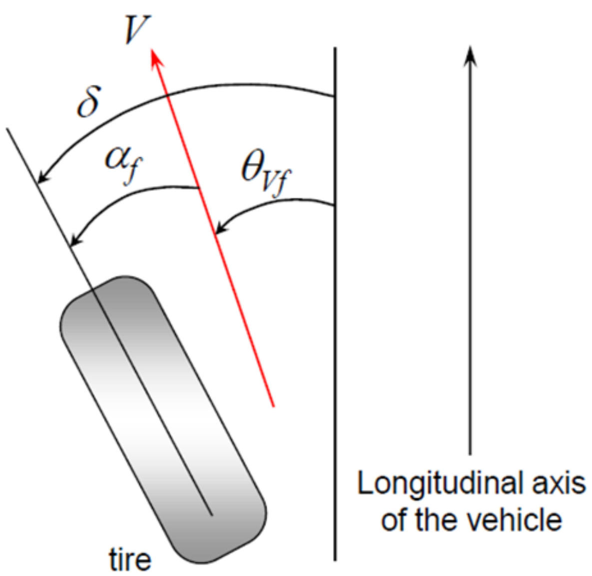
$$\alpha_f = \delta - \theta_{Vf}$$

θ_{Vf} : angle that the velocity vector makes with the longitudinal axis of the vehicle

δ : front steering angle

  - Slip angle of the rear wheel

$$\alpha_r = -\theta_{Vr}$$



Bicycle Model of Lateral Vehicle Dynamics

- Lateral tire force for the front wheel

$$F_{yf} = 2C_{\alpha f}(\delta - \theta_{Vf})$$

  - Proportionality constant $C_{\alpha f}$ is called the cornering stiffness of each front tire
  - The factor 2 accounts for two front wheels

- Lateral tire force for the rear wheel

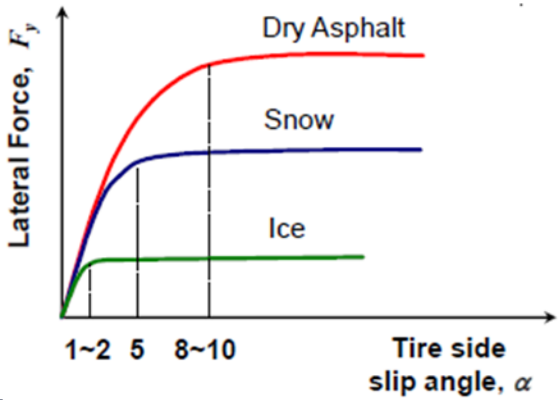
$$F_{yr} = 2C_{\alpha r}(-\theta_{Vr})$$

- The slope at low slip values is called the cornering stiffness C_{α}

$$F_y = C_\alpha \alpha, \quad |\alpha| < 5 \text{ deg}$$

  - However, the tire model will be *no longer linear at large slip angles !*

  ➔ *Lateral tire force* $= f(F_z, \mu, \alpha, \cdots)$



Bicycle Model of Lateral Vehicle Dynamics

- Tire slip angle calculation

$$\tan\theta_{Vf} = \frac{V_y + l_f\dot{\psi}}{V_x} \qquad \tan\theta_{Vr} = \frac{V_y - l_r\dot{\psi}}{V_x}$$

- Using small angle assumption ($\tan\theta \simeq \theta$) and the notation $V_y = \dot{y}$

$$\theta_{Vf} = \frac{\dot{y} + l_f\dot{\psi}}{V_x} \qquad \alpha_f = \delta - \theta_{Vf}$$

$$\theta_{Vr} = \frac{\dot{y} - l_r\dot{\psi}}{V_x} \qquad \alpha_r = -\theta_{Vr}$$

Bicycle Model of Lateral Vehicle Dynamics

- State space model

$$\begin{bmatrix} \dot{y} \\ \ddot{y} \\ \dot{\psi} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{2C_{\alpha f}+2C_{\alpha r}}{mV_x} & 0 & -V_x - \frac{2l_f C_{\alpha f}-2l_r C_{\alpha r}}{mV_x} \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{2l_f C_{\alpha f}-2l_r C_{\alpha r}}{I_z V_x} & 0 & -\frac{2l_f^2 C_{\alpha f}+2l_r C_{\alpha r}}{I_z V_x} \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \\ \psi \\ \dot{\psi} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ \frac{2C_{\alpha f}}{m} \\ 0 \\ \frac{2l_f C_{\alpha f}}{I_z} \end{bmatrix} \delta$$

$$\Longrightarrow \dot{x} = Ax + Bu$$