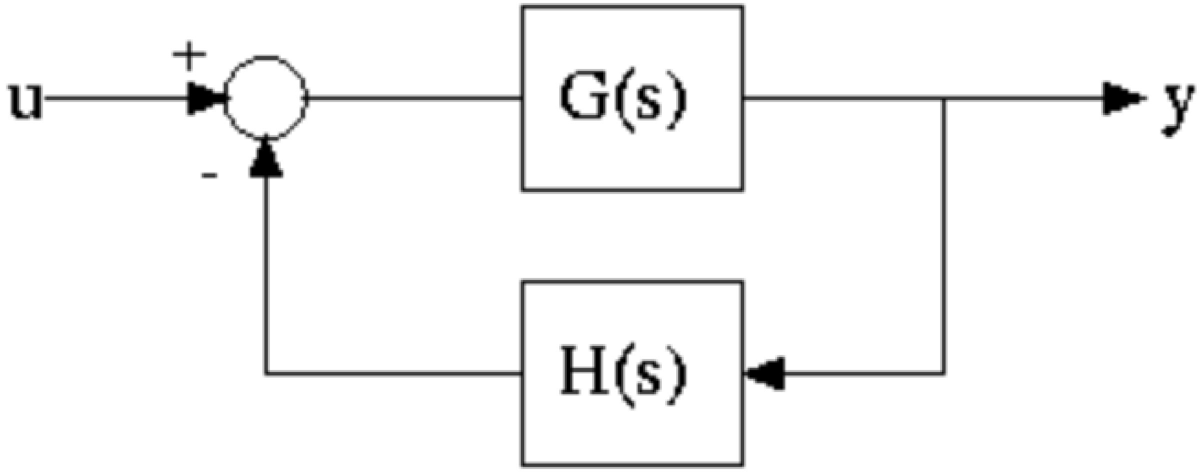
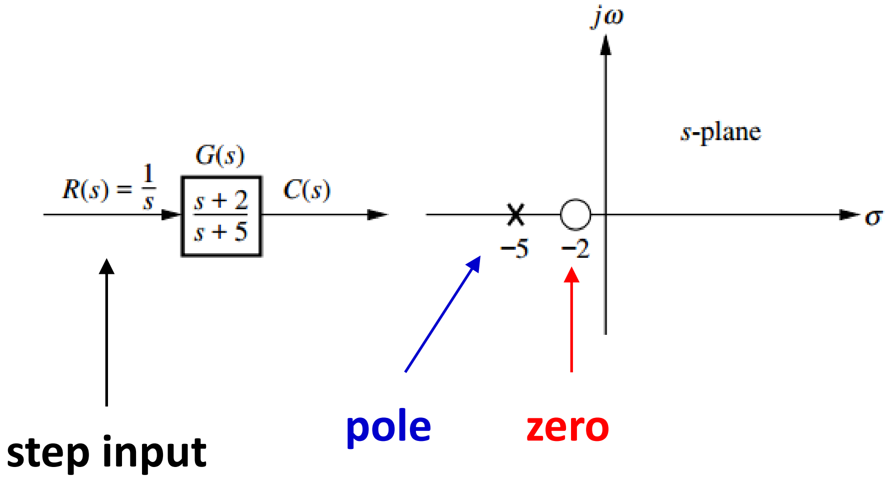
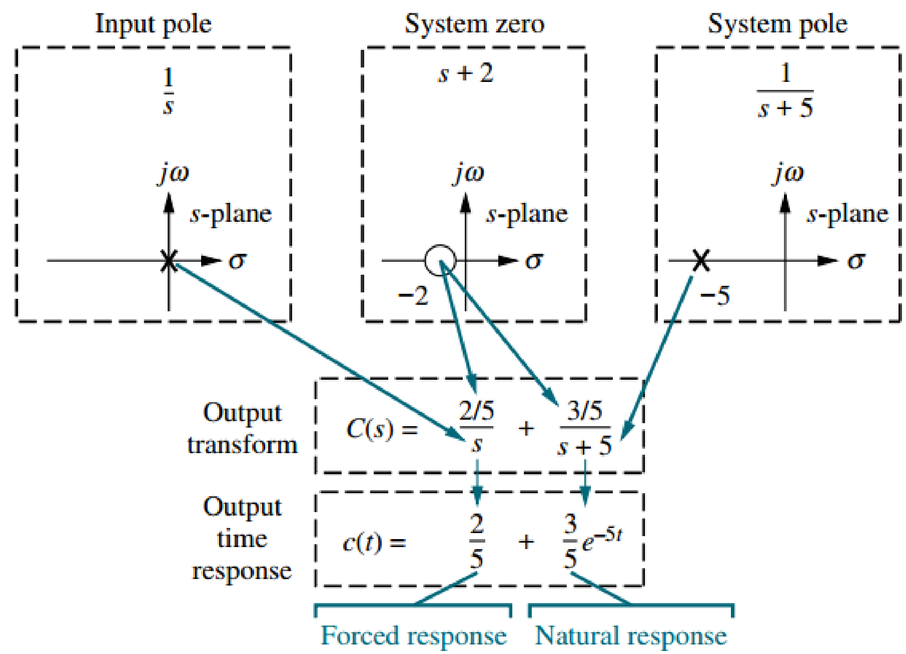
System Control

time response



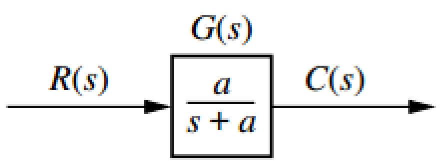
Poles, zeros, and system response

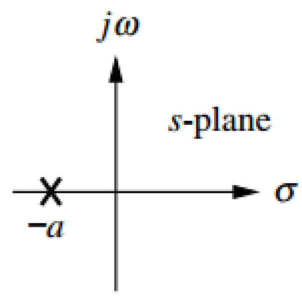




Name	$f(t)$	$F(s)$
Impulse	$f(t) = \begin{cases} 1 & t = 0 \\ 0 & t > 0 \end{cases}$	1
Step	$f(t) = 1$	$\frac{1}{s}$
Ramp	$f(t) = t$	$\frac{1}{s^2}$

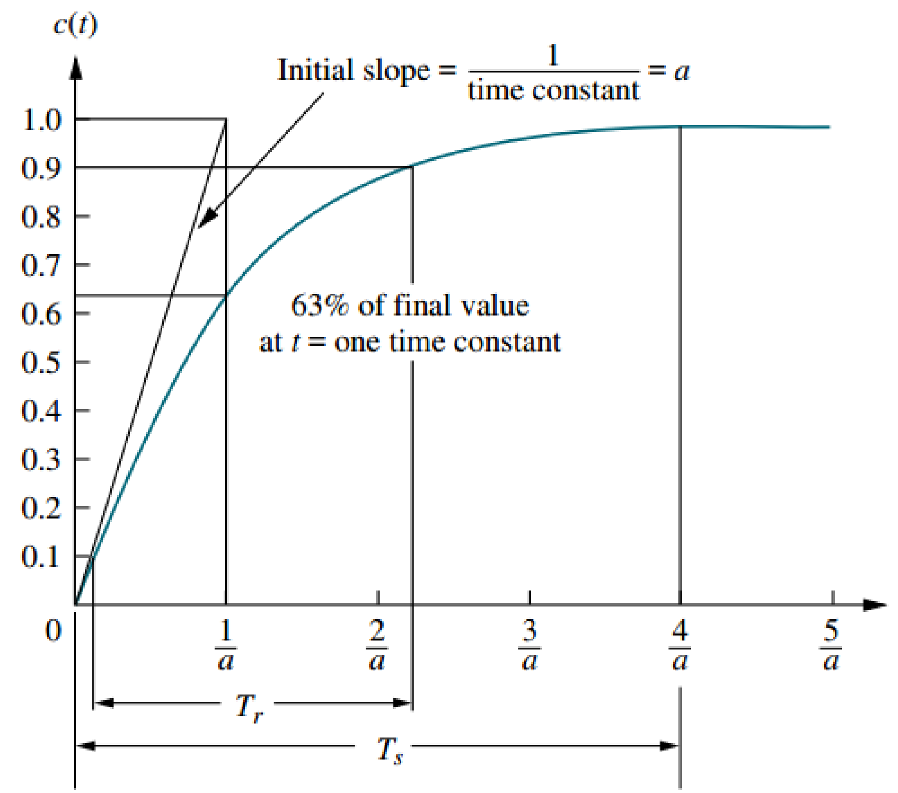
First-order system





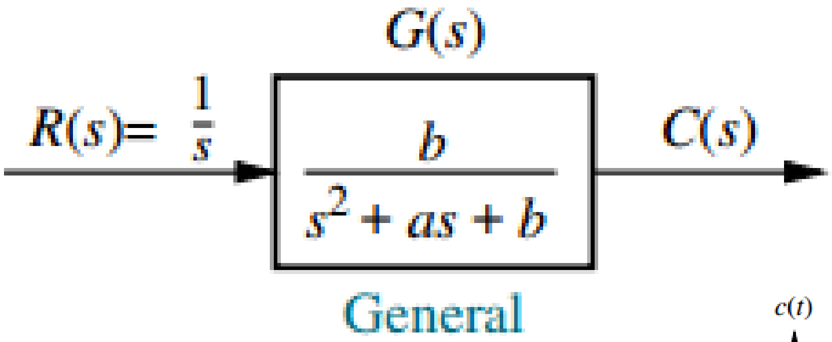
$$C(s) = R(s)G(s) = \frac{a}{s(s+a)}$$

$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$



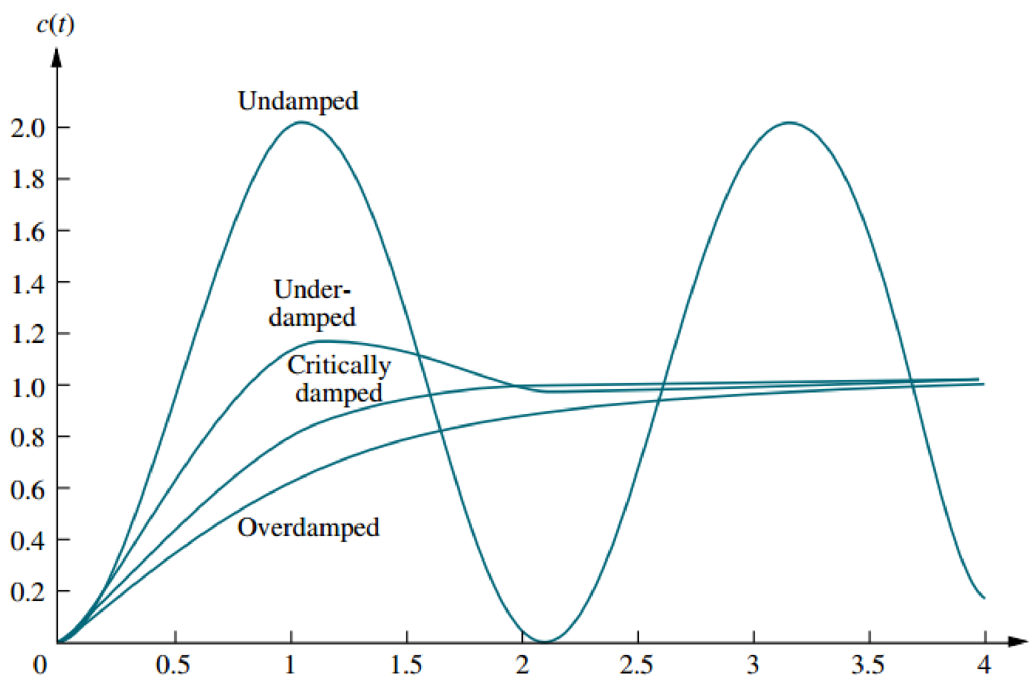
1. Time constant
2. Rise time
3. Settling time

Second-order system (1)



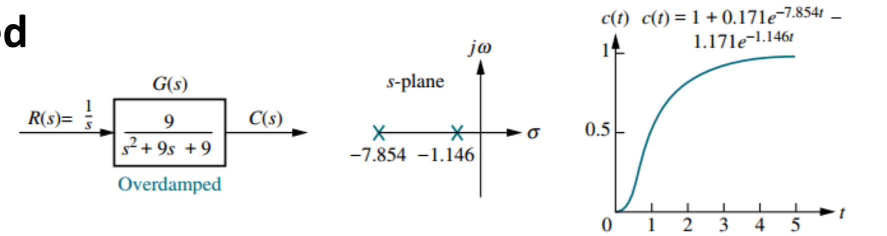
$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

1. Natural Frequency
2. Damping ratio
3. Peak time
4. Percent overshoot
5. Rise time
6. Settling time



Second-order system (1)

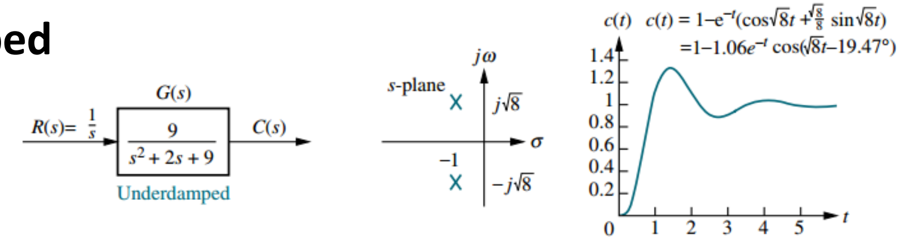
Overdamped



Poles: Two real at $-\sigma_1,\ -\sigma_2$

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$$

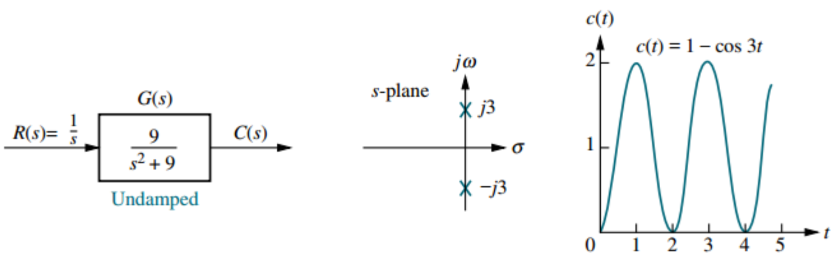
Underdamped



Poles: Two complex at $-\sigma_d \pm j\omega_d$

$$c(t) = Ae^{-\sigma_d t}\cos(\omega_d t - \phi)$$

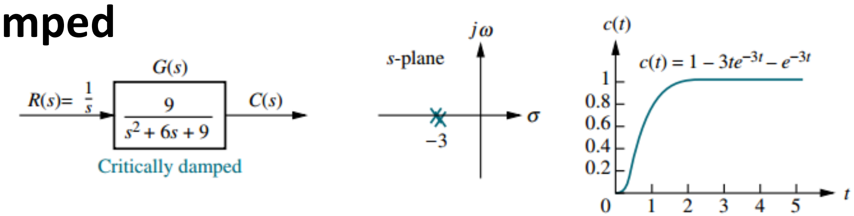
Undamped



Poles: Two imaginary at $\pm j\omega_1$

$$c(t) = \mathrm{A}\cos(\omega_1 t - \phi)$$

Critically damped



Poles: Two real at $-\sigma_1$

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 t e^{-\sigma_1 t}$$

General Second-order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

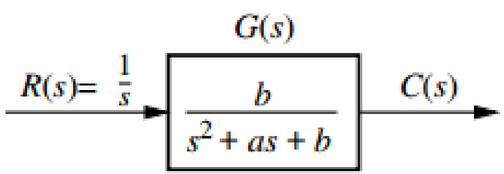
$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

- **Natural frequency**

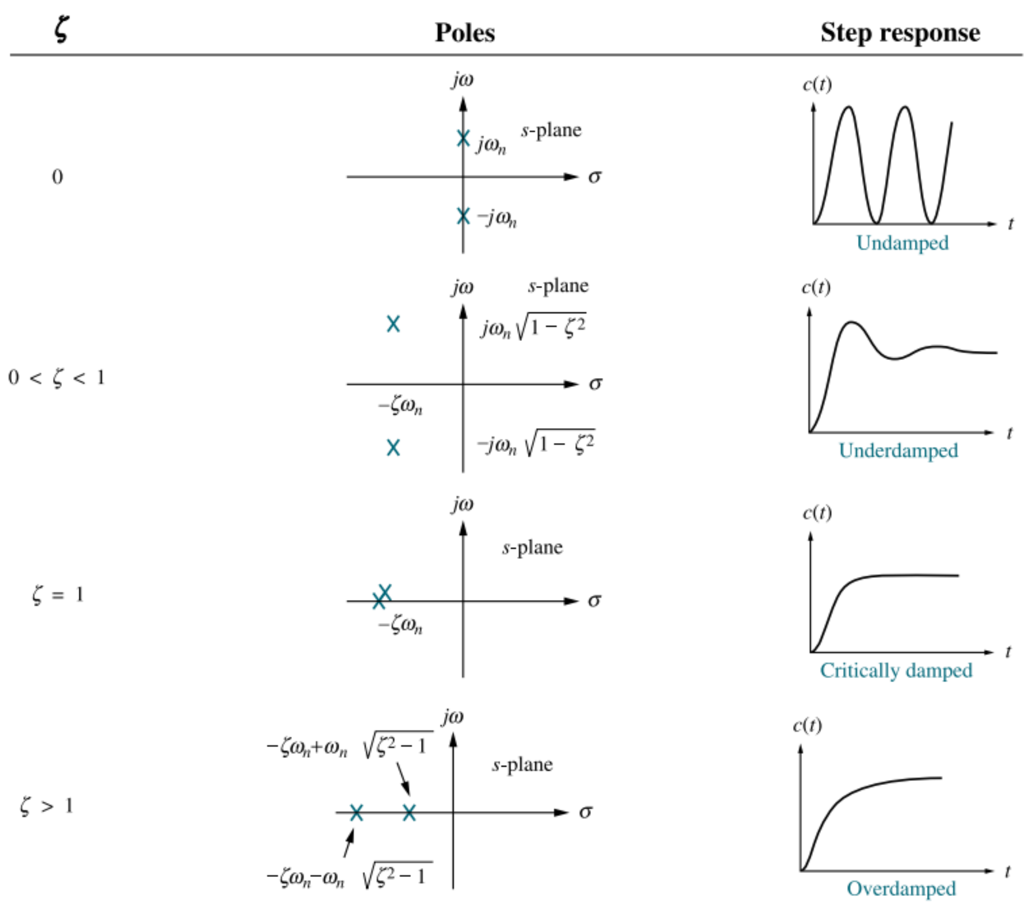
$$G(s) = \frac{b}{s^2 + b}$$

$$\omega_n = \sqrt{b}$$

- **Damping ratio**



$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency (rad/second)}} = \frac{|\sigma|}{\omega_n} = \frac{a/2}{\omega_n}$$



Example : second-order system

Finding ζ and ω_n For a Second-Order System

$$G(s) = \frac{400}{s^2 + 12s + 400}$$

$$G(s) = \frac{225}{s^2 + 30s + 225}$$

$$G(s) = \frac{900}{s^2 + 90s + 900}$$

$$G(s) = \frac{625}{s^2 + 625}$$

$$G(s) = \frac{36}{s^2 + 4.2s + 36}$$

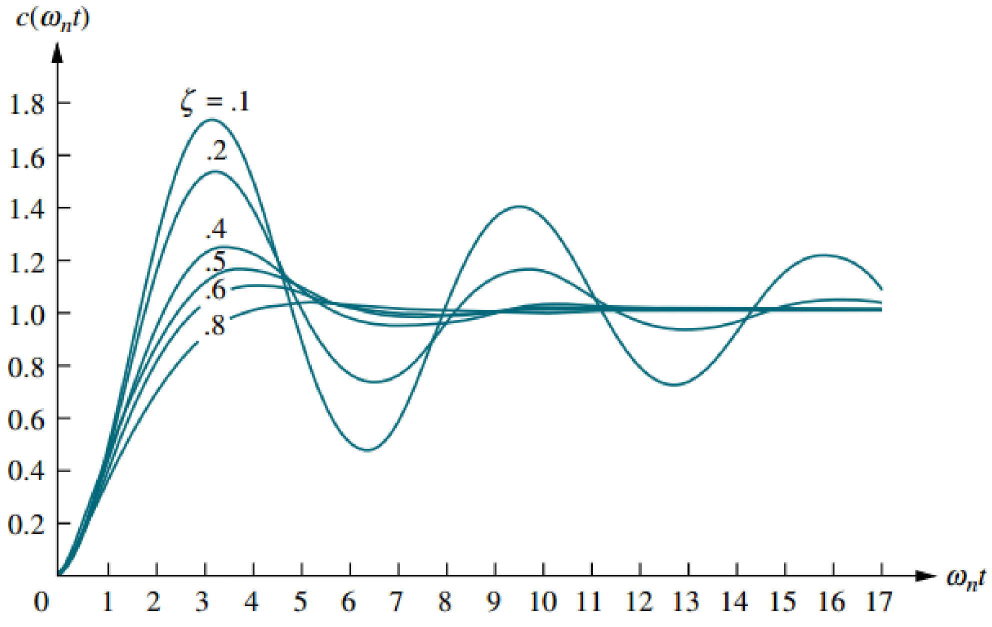
Underdamped second-order systems (1)

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$= \frac{1}{s} - \frac{(s + \zeta\omega_n) + \frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

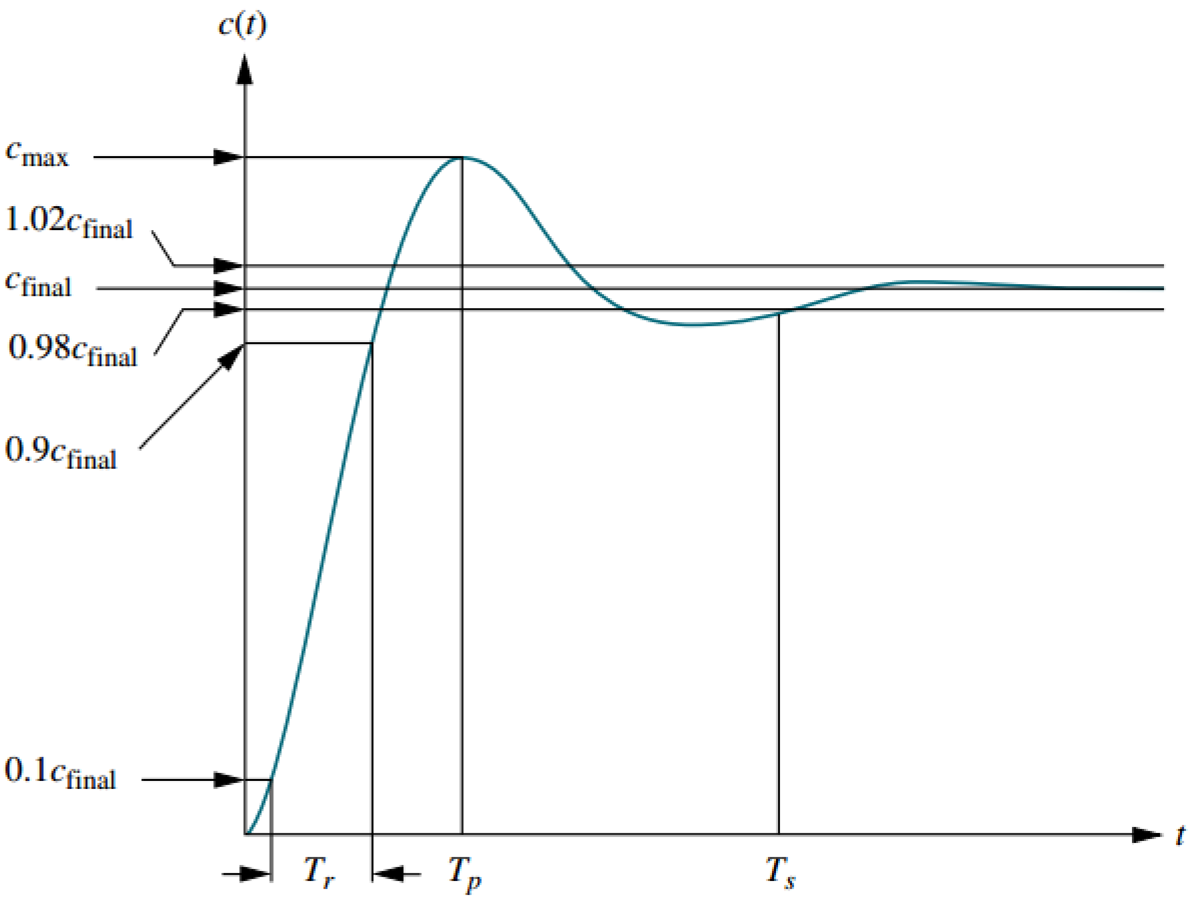
$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_n\sqrt{1-\zeta^2}t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_n\sqrt{1-\zeta^2}t\right)$$

$$= 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos(\omega_n\sqrt{1-\zeta^2}t - \phi)$$



Underdamped second-order systems (2)

1. Peak time
2. Percent overshoot
3. Rise time
4. Settling time



Peak time

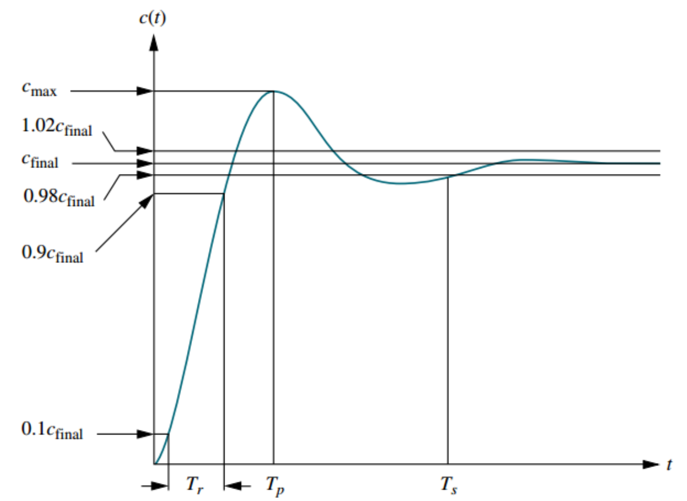
$$\mathscr{L}[\dot{c}(t)] = sC(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

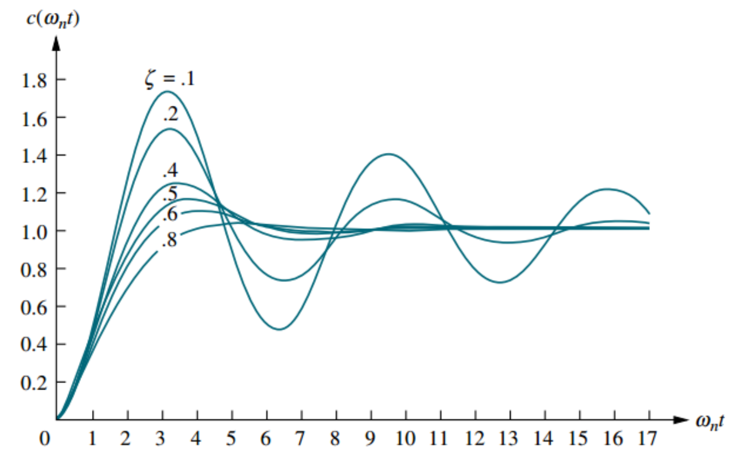
$$= \frac{\omega_n^2}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} = \frac{\frac{\omega_n}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

$$\dot{c}(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t$$

$$\omega_n\sqrt{1-\zeta^2}\, t = n\pi \quad \text{or} \quad t = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$





%OS

$$\%OS = \frac{c_{\max} - c_{\text{final}}}{c_{\text{final}}} \times 100$$

$$c_{\max} = c(T_p) = 1 - e^{-(\zeta\pi/\sqrt{1-\zeta^2})}\left(\cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi\right)$$

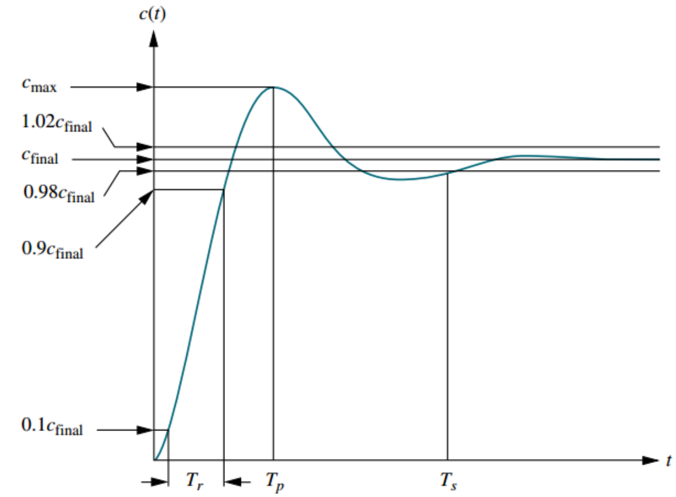
$$= 1 + e^{-(\zeta\pi/\sqrt{1-\zeta^2})}$$

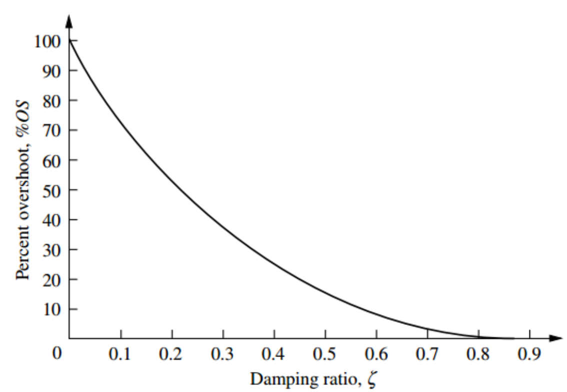
and

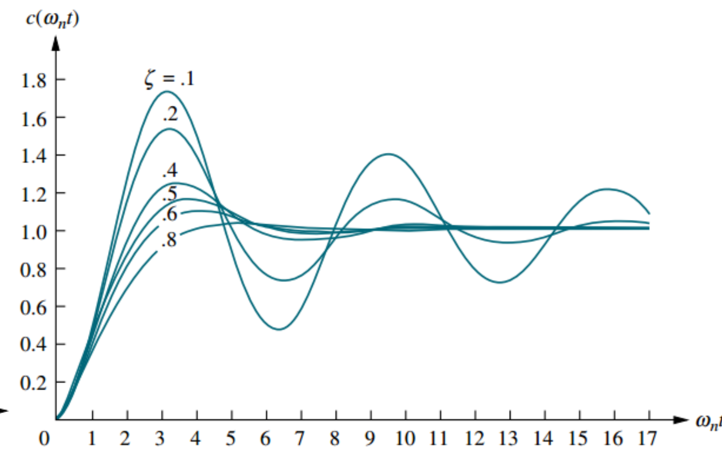
$$c_{\text{final}} = 1$$

$$\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100$$

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$







Settling time & Rise time

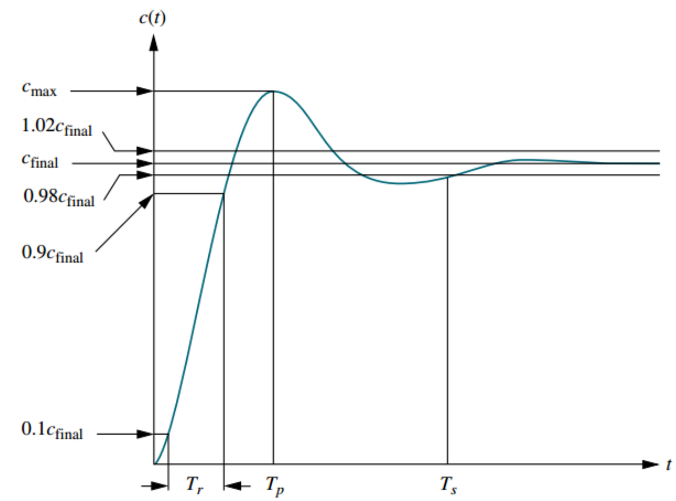
- Settling time

$$e^{-\zeta\omega_n t}\frac{1}{\sqrt{1-\zeta^2}} = 0.02$$

$$T_s = \frac{-\ln(0.02\sqrt{1-\zeta^2})}{\zeta\omega_n}$$

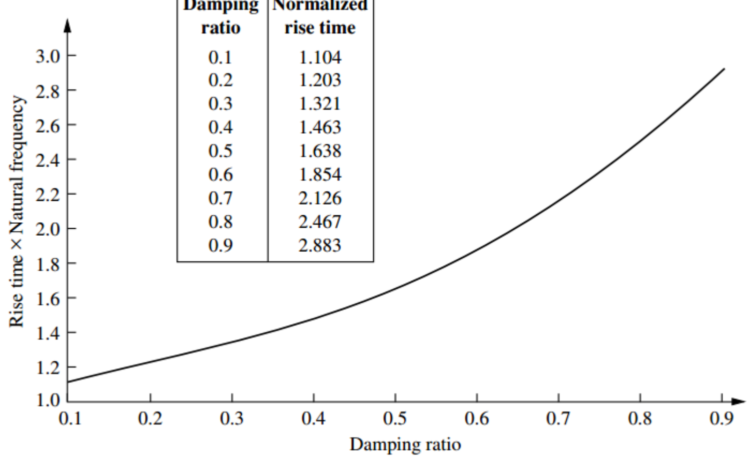
ζ varies from 0 to 0.9.

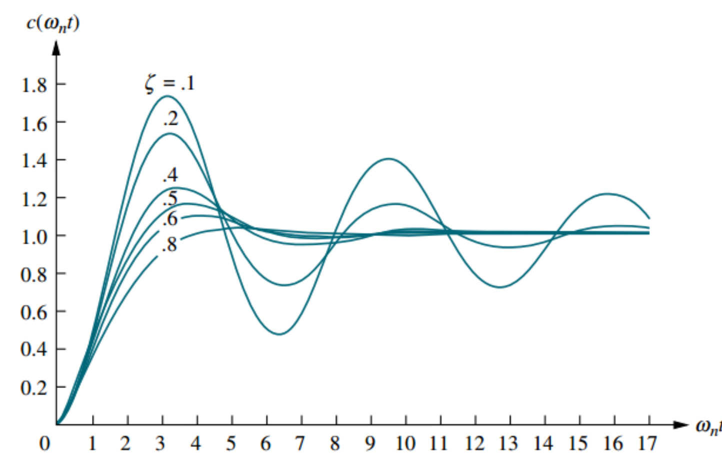
$$T_s = \frac{4}{\zeta\omega_n}$$



- Rise time

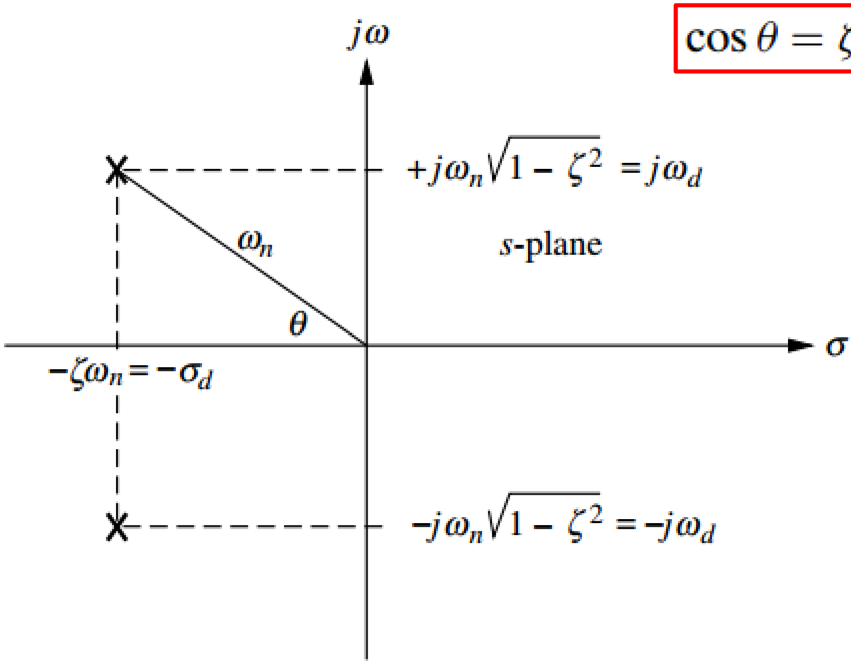
Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883

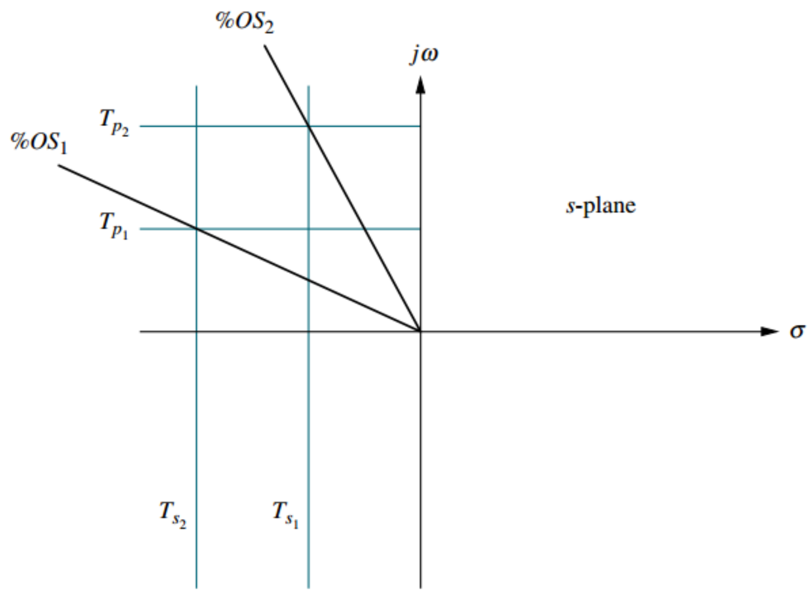




%OS & peak time from pole location

$$\cos\theta = \zeta.$$





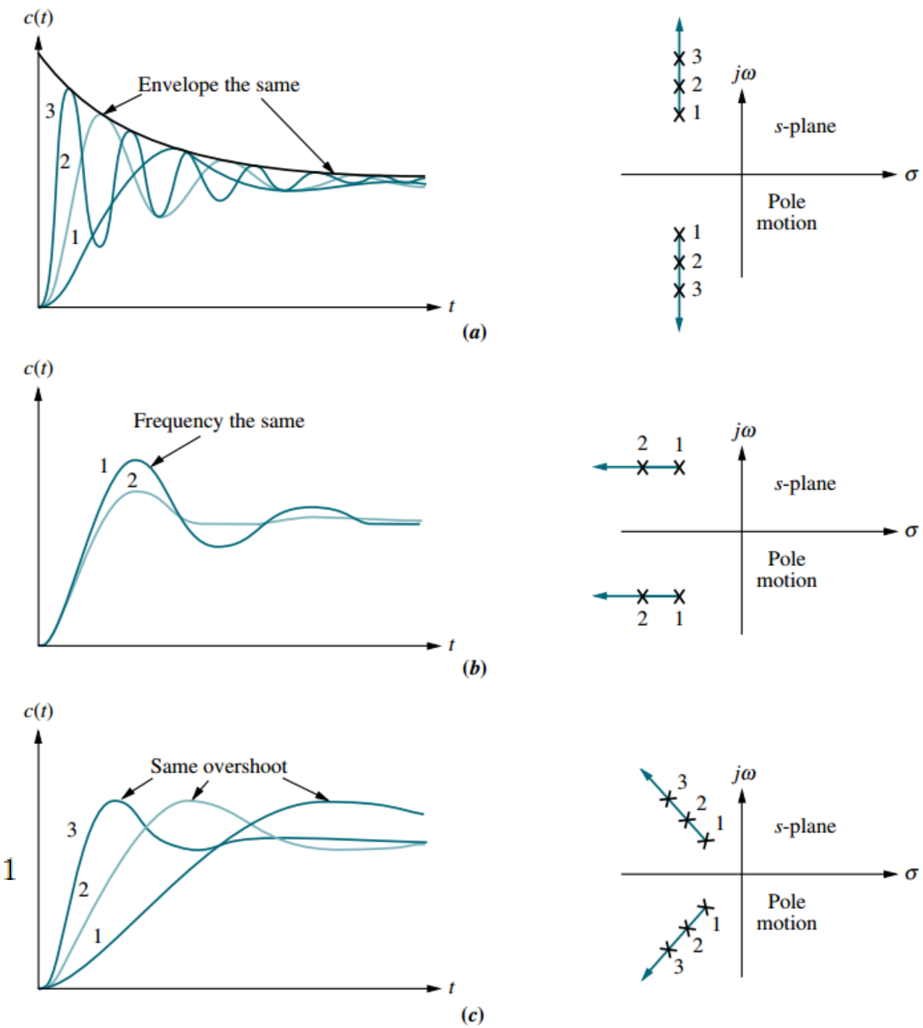
$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

$$T_s = \frac{4}{\zeta\omega_n} = \frac{\pi}{\sigma_d}$$

Characteristics of second-order system

$$\cos\theta = \zeta.$$

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



1. Natural Frequency: ω_n
2. Damping ratio: ζ
3. Peak time: $T_p = \dfrac{\pi}{\omega_d}$
4. Percent overshoot:
$$\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100$$
5. Rise time
$$\omega_n T_r = 1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1$$
6. Settling time: $T_s = \dfrac{4}{\sigma_d}$