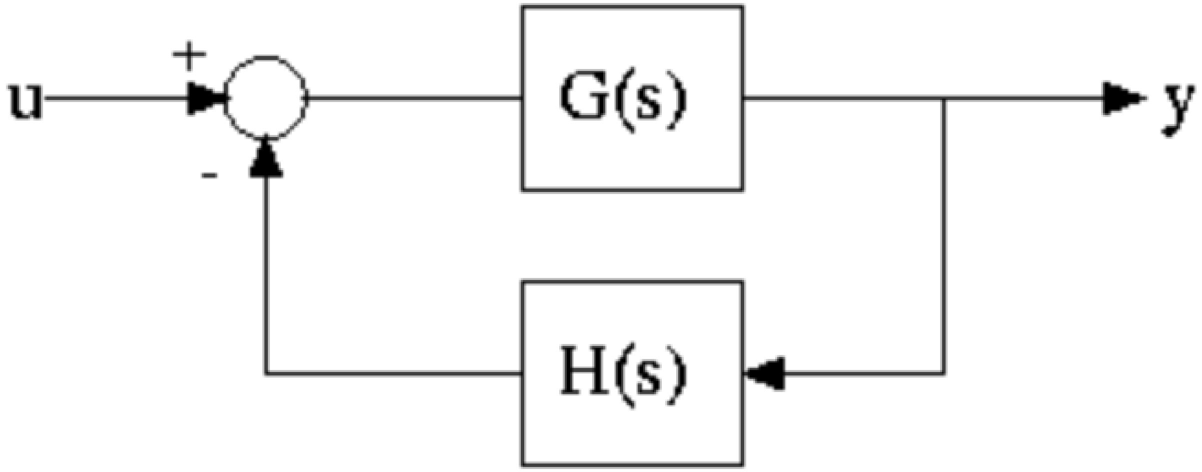
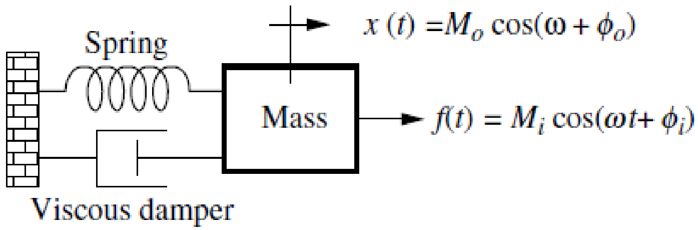
System Control

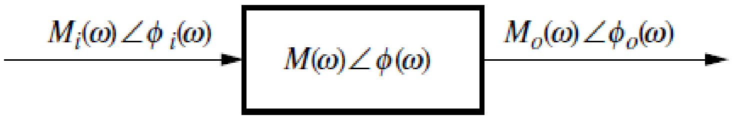
Frequency Response Techniques



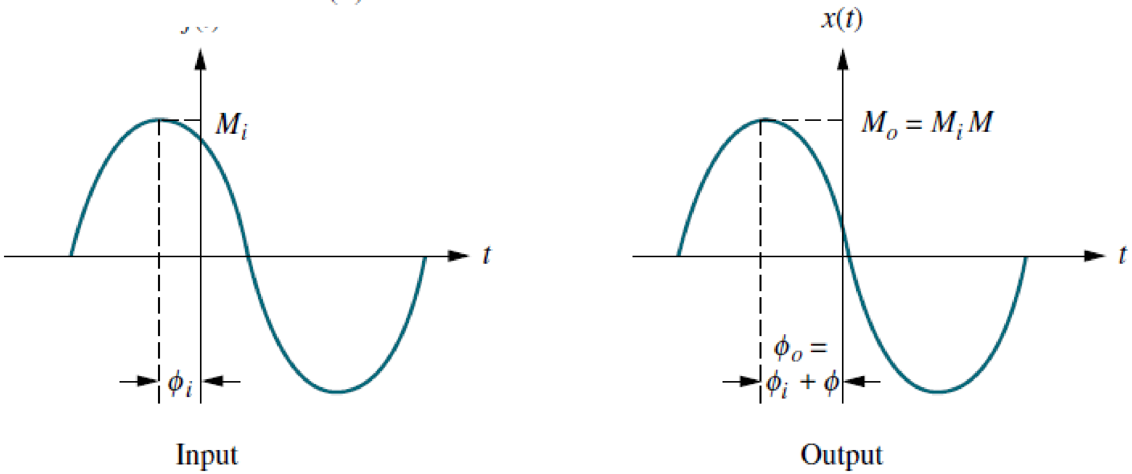
The concept of frequency response



(*a*)



(*b*)



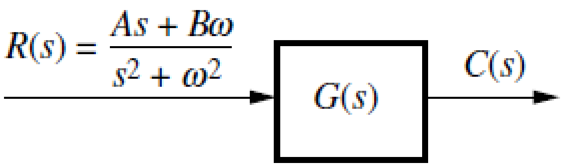
(*c*)

$$M_o(\omega)\angle\phi_o(\omega) = M_i(\omega)M(\omega)\angle[\phi_i(\omega) + \phi(\omega)]$$

$$M(\omega) = \frac{M_o(\omega)}{M_i(\omega)}$$

$$\phi(\omega) = \phi_o(\omega) - \phi_i(\omega)$$

Magnitude & Phase



Laplace transform of a general sinusoid,

$$r(t) = A\cos\omega t + B\sin\omega t = \sqrt{A^2 + B^2}\cos\left[\omega t - \tan^{-1}(B/A)\right]$$

in polar form, $M_i \angle \phi_i$, where $M_i = \sqrt{A^2 + B^2}$ and $\phi_i = -\tan^{-1}(B/A)$

$$C(s) = \frac{As + B\omega}{(s^2 + \omega^2)} G(s)$$

Magnitude: $|G(j\omega)| = \log|G_1 \; G_2 \cdots\cdots G_n| = \log|G_1| + \cdots\cdots + \log|G_n|$

Phase: $\angle G(j\omega) = \angle(G_1 \; G_2 \cdots\cdots G_n) = \angle G_1 + \cdots\cdots + \angle G_n$

Bode plot

- A **Bode plot** is a (semilog) plot of the transfer function magnitude and phase angle as a function of frequency
- The gain magnitude is many times expressed in terms of decibels (dB)

$$\text{dB} = 20\log_{10} A = 20\log_{10}\frac{V_{out}}{V_{in}}$$

where A is the amplitude or gain

- a *decade* is defined as any 10-to-1 frequency range
- an *octave* is any 2-to-1 frequency range

20 dB/decade = 6 dB/octave

$$3\text{dB} = 20\log_{10}\sqrt{2}$$

$$-3\text{dB} = 20\log_{10}\frac{1}{\sqrt{2}}$$

$$10\text{dB} = 20\log_{10}\sqrt{10}$$

$$20\text{dB} = 20\log_{10} 10$$

Improving steady-state error via cascade compensation

1. Constant gain, $G(s) = K$

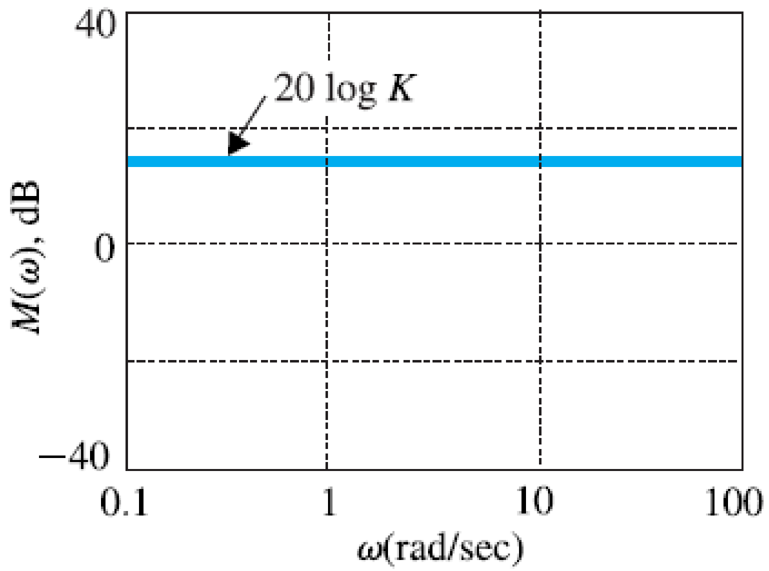
2. Differential and integral $G(s) = s$ & $1/s$

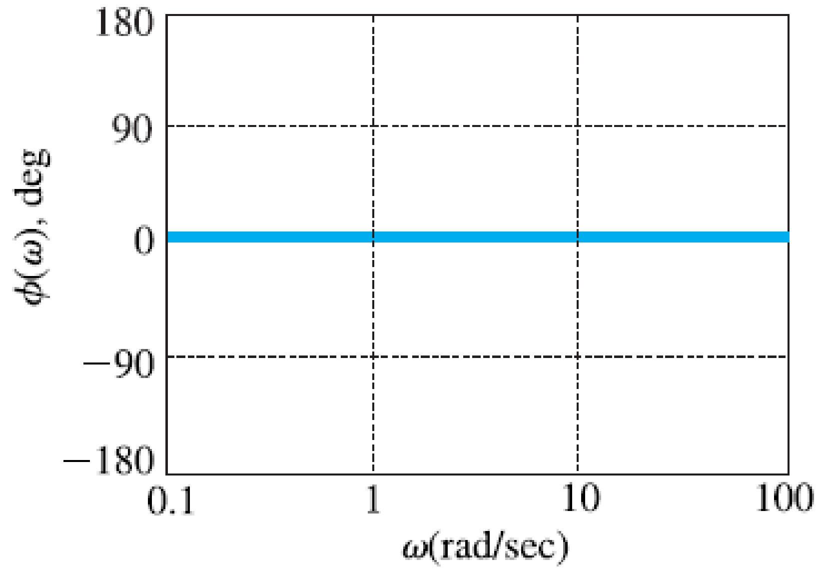
3. $G(s) = 1/(1+Ts)$ & $1+Ts$

4. $G(s) = 1/(1+\frac{2\zeta}{\omega_n}s+\frac{1}{\omega_n^2}s^2)$ & $1+\frac{2\zeta}{\omega_n}s+\frac{1}{\omega_n^2}s^2$

5. $G(s) = e^{-Ts}$

Constant gain





Differential and integral (1)

- **Differential:** $[G(s) = s]$, $M(\omega)$ & $\phi(\omega)$

$$M(\omega) = 20 \log|j\omega| = 20 \log \omega \text{ dB}$$

$$\phi(\omega) = \tan^{-1}\infty = 90^\circ$$

- **Integral:** $[G(s) = 1/s]$, $M(\omega)$ & $\phi(\omega)$

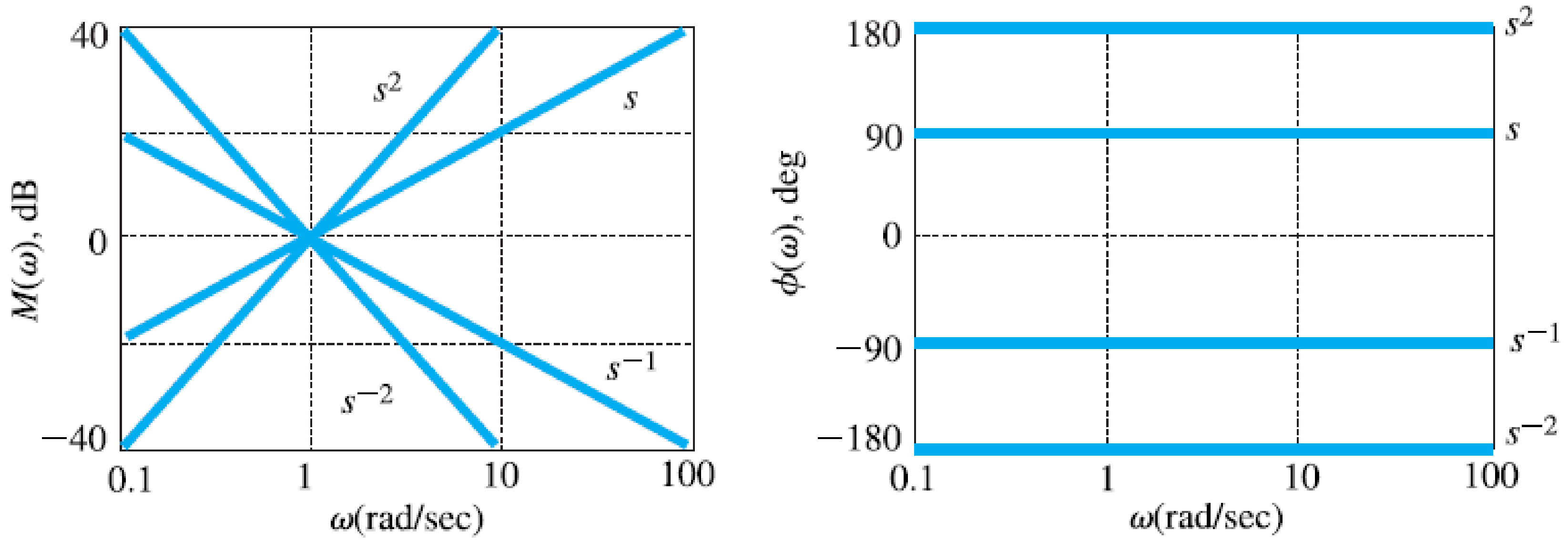
$$M(\omega) = 20\log\left|\frac{1}{j\omega}\right| = -20\log\omega \text{ dB}$$

$$\phi(\omega) = -\tan^{-1}\infty = -90^\circ$$

Differential and integral (2)

$$M(\omega) = 20n \log \omega \text{ dB}$$

$$\phi(\omega) = 90n^\circ$$



Real axis pole $H(s) = \frac{a}{s+a}$

$$G(s) = \frac{1}{(s+a)} = \frac{1}{a\left(\frac{s}{a}+1\right)}$$

At low frequencies when ω approaches zero,

The magnitude response in dB is

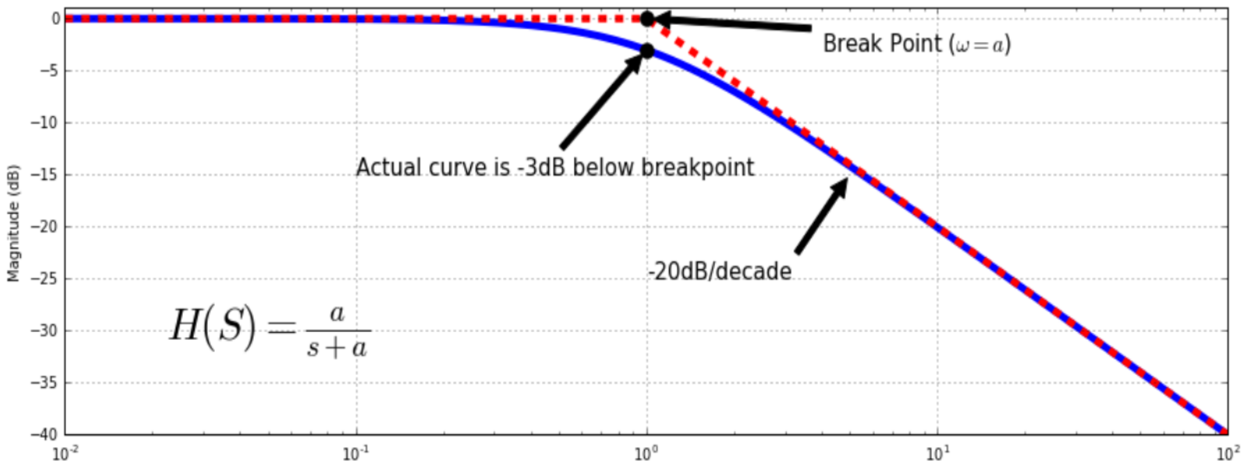
$$20 \log M = 20 \log (1/a)$$

At high frequencies where $\omega \gg a$,

$$G(j\omega) = \frac{1}{a\left(\frac{s}{a}\right)}\bigg|_{s \to j\omega} = \frac{1}{a\left(\frac{j\omega}{a}\right)} = \frac{\frac{1}{a}}{\frac{\omega}{a}}\angle -90^\circ = \frac{1}{\omega}\angle -90^\circ$$

$$20 \log M = 20 \log \frac{1}{a} - 20 \log \frac{\omega}{a} = -20 \log \omega$$

Real axis pole $H(s) = \frac{a}{s+a}$

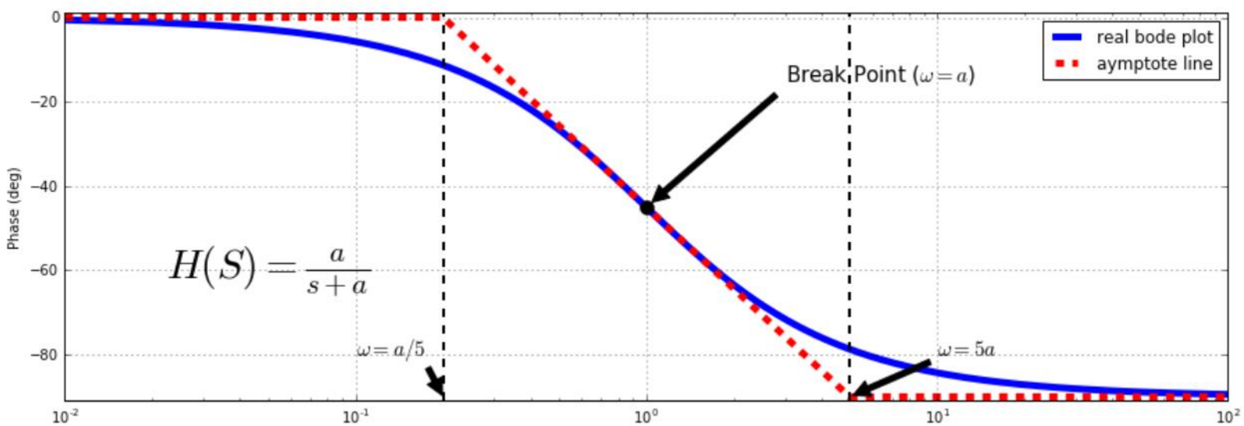


Breakpoint : $\omega = a$

Low Freq. asymptote 0dB

High Freq. asymptote -20dB/decade

Actual curve is -3dB below breakpoint



Low Freq. asymptote $0°$

$-45°$ at breakpoint $\omega = a$

High Freq. asymptote $-90°$

$0°$ at $\omega \approx \frac{a}{5}$, $-90°$ at $\omega \approx 5a$

Real axis zero $H(s) = \frac{s+b}{b}$

$$G(j\omega) = (j\omega + a) = a\left(j\frac{\omega}{a} + 1\right)$$

At low frequencies when ω approaches zero,

$$G(j\omega) \approx a$$

The magnitude response in dB is

$$20 \log M = 20 \log a$$

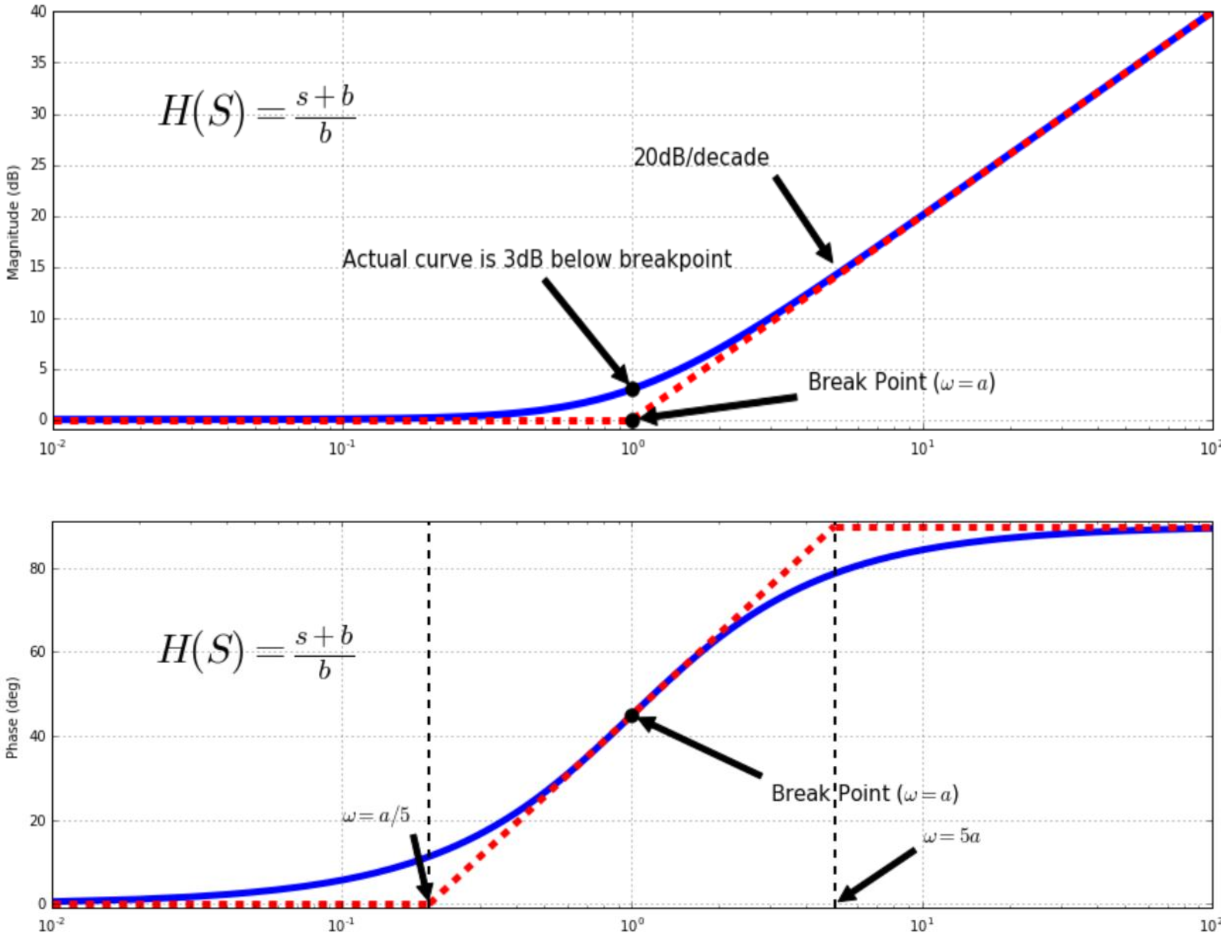
At high frequencies where $\omega \gg a$,

$$G(j\omega) \approx a\left(\frac{j\omega}{a}\right) = a\left(\frac{\omega}{a}\right)\angle 90^\circ = \omega \angle 90^\circ$$

The magnitude response in dB is

$$20 \log M = 20 \log a + 20 \log \frac{\omega}{a} = 20 \log \omega$$

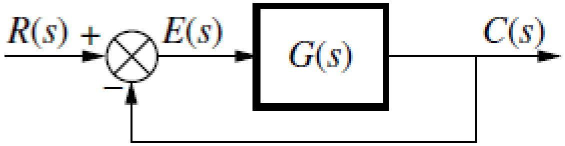
Real axis zero $H(s) = \dfrac{s+b}{b}$



Breakpoint : $\omega = b$
Low Freq. asymptote 0dB
High Freq. asymptote 20dB/decade
Actual curve is 3dB below breakpoint

Low Freq. asymptote $0°$
$45°$ at breakpoint $\omega = b$
High Freq. asymptote $90°$
$0°$ at $\omega \approx \frac{a}{5}$, $90°$ at $\omega \approx 5a$

Example 10.2 : Magnitude

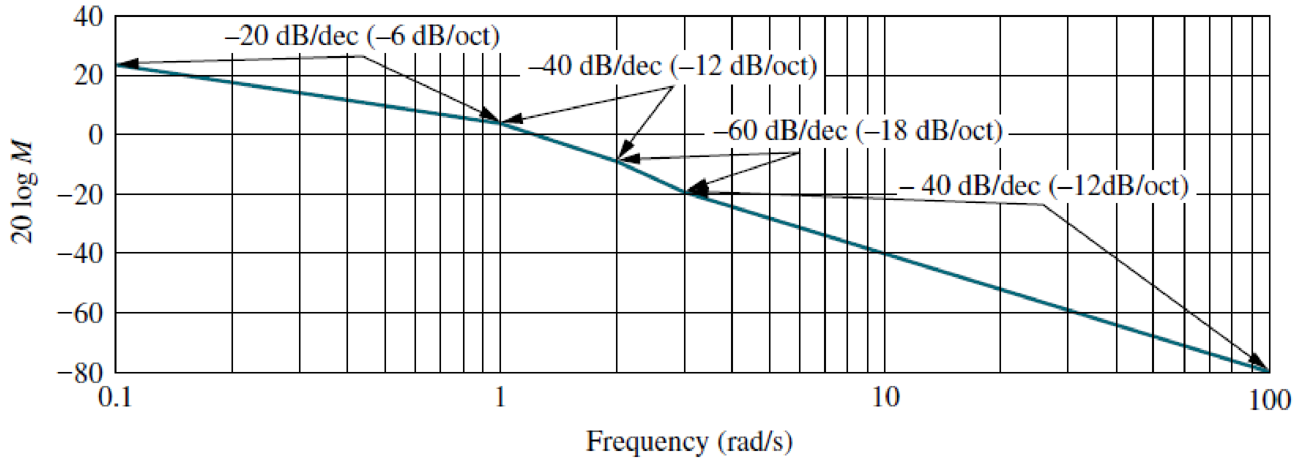


PROBLEM: Draw the Bode plots for the system $G(s) = K(s+3)/[s(s+1)(s+2)]$

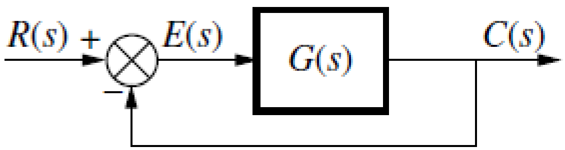
break frequencies are at 1, 2, and 3.

$$G(s) = \frac{\frac{3}{2}K\left(\frac{s}{3}+1\right)}{s(s+1)\left(\frac{s}{2}+1\right)}$$

Description	Frequency (rad/s) 0.1 (Start: Pole at 0)	1 (Start: Pole at −1)	2 (Start: Pole at −2)	3 (Start: Zero at −3)
Pole at 0	−20	−20	−20	−20
Pole at −1	0	−20	−20	−20
Pole at −2	0	0	−20	−20
Zero at −3	0	0	0	20
Total slope (dB/dec)	−20	−40	−60	−40

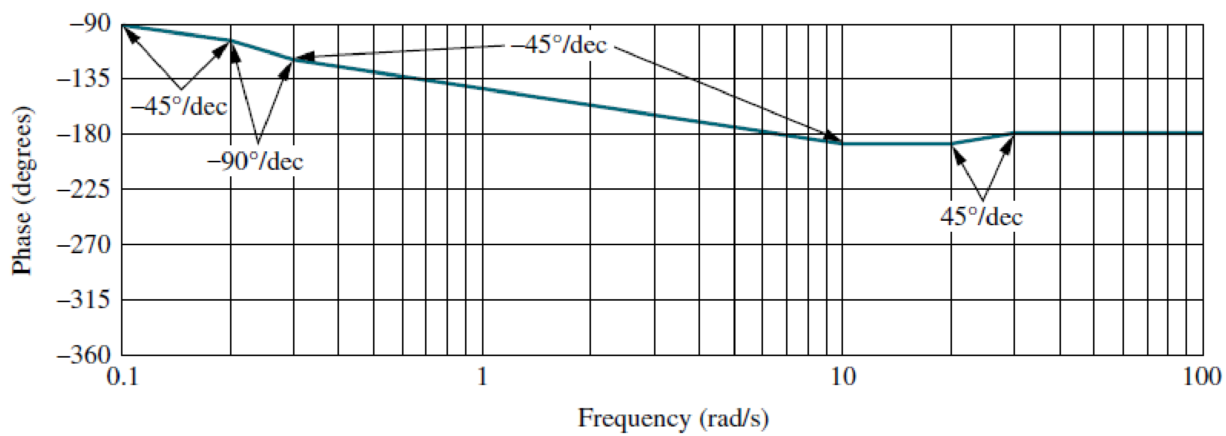


Example 10.2 : Phase



PROBLEM: Draw the Bode plots for the system $G(s) = K(s+3)/[s(s+1)(s+2)]$

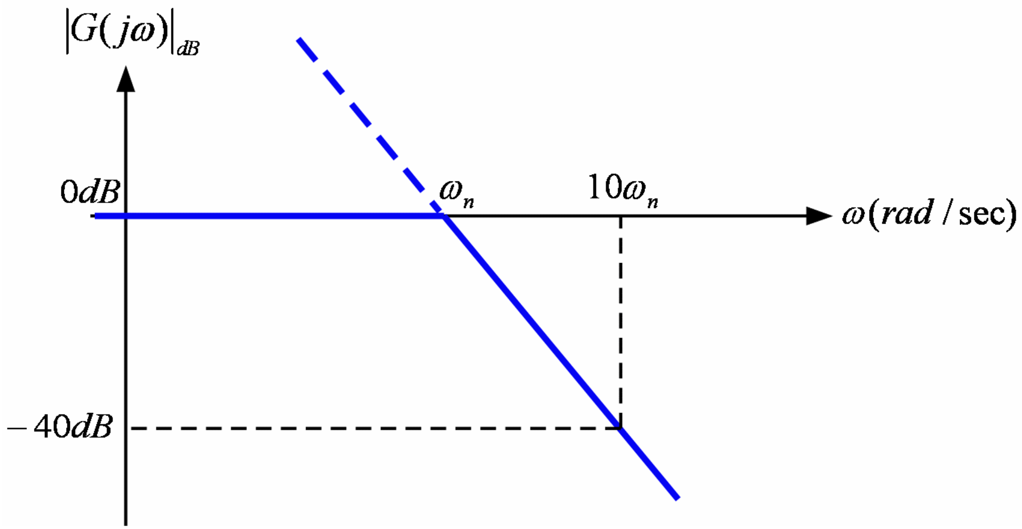
	Frequency (rad/s)					
Description	**0.1 (Start: Pole at –1)**	**0.2 (Start: Pole at –2)**	**0.3 (Start: Pole at –3)**	**0 (End: Pole at –1)**	**20 (End: Pole at –2)**	**30 (End: Zero at –3)**
Pole at –1	–45	–45	–45	0		
Pole at –2		–45	–45	–45	0	
Zero at –3			45	45	45	0
Total slope (deg/dec)	–45	–90	–45	0	45	0



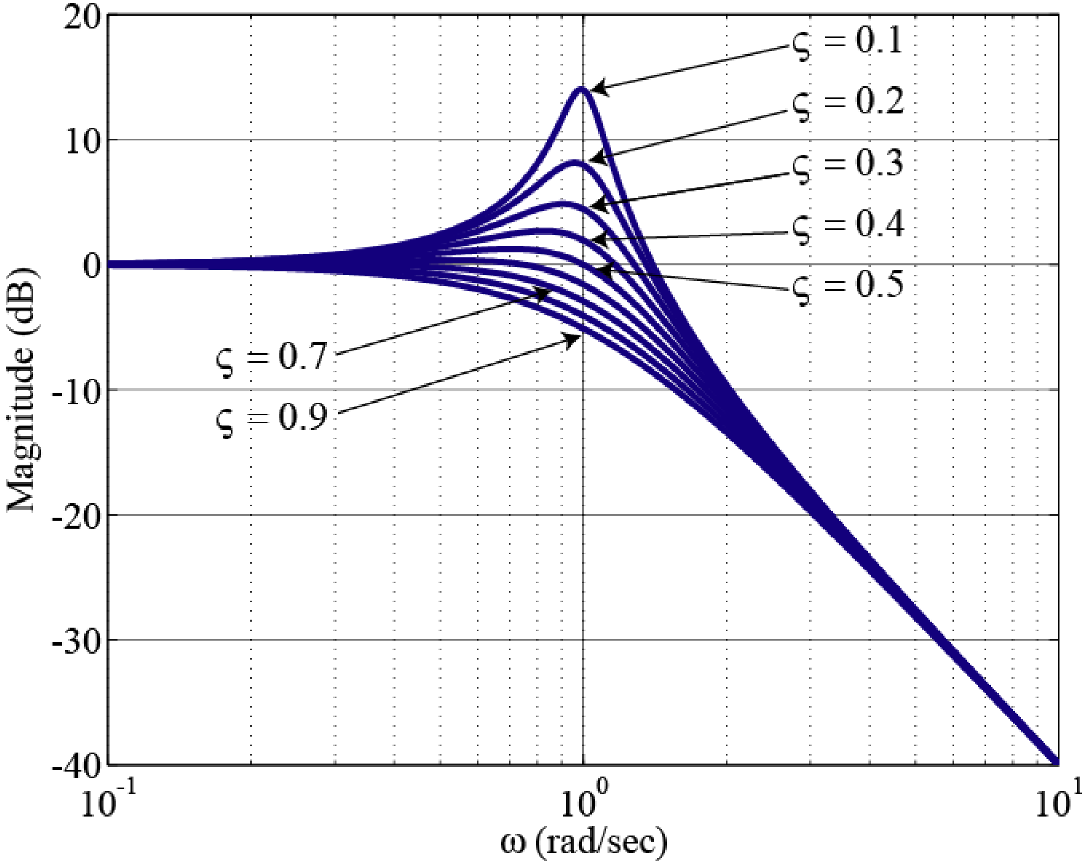
Complex conjugate poles $1/\left[\left(s/\omega_n\right)^2+\left(2\zeta s/\omega_n\right)+1\right]$

$$\left|\frac{1}{\left(j\omega/\omega_n\right)^2+\left(2\varsigma j\omega/\omega_n\right)+1}\right|_{dB} \approx -20\log_{10}1=0 \qquad \omega \ll \omega_n$$

$$\left|\frac{1}{\left(j\omega/\omega_n\right)^2+\left(2\varsigma j\omega/\omega_n\right)+1}\right|_{dB} \approx -20\log_{10}\left|\left(\omega/\omega_n\right)^2\right| \qquad \omega \gg \omega_n$$



Complex conjugate poles $1/\left[(s/\omega_n)^2 + (2\zeta s/\omega_n) + 1\right]$



$$\left|\frac{1}{(j\omega/\omega_n)^2 + (2\zeta j\omega/\omega_n) + 1}\right|_{\omega=\omega_n} = \frac{1}{2\zeta}$$

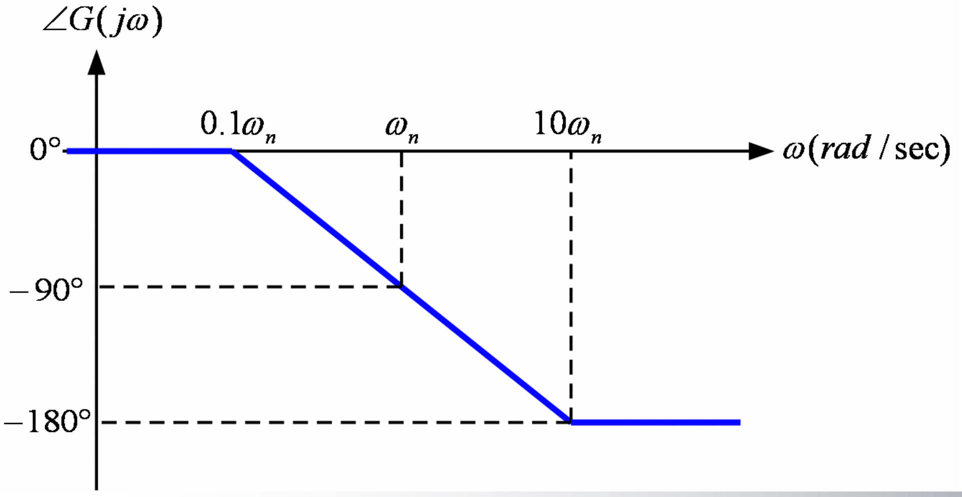
- Resonance peak
- Cutoff frequency
- Bandwidth

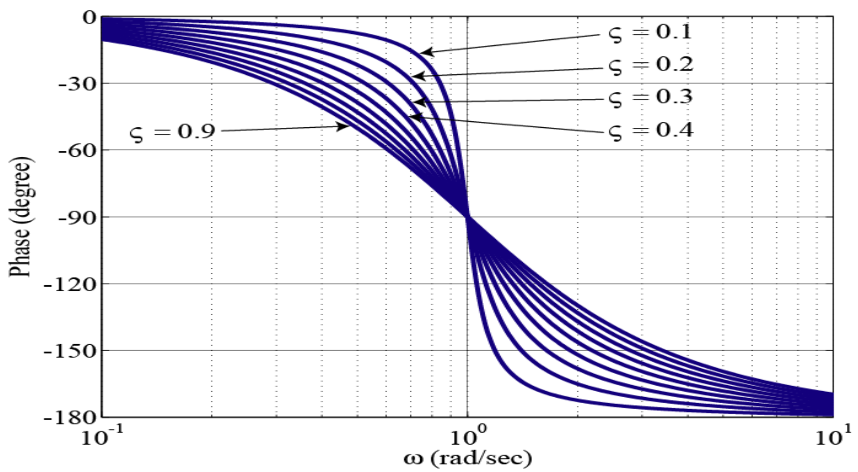
Complex conjugate poles $1/\left[\left(s/\omega_n\right)^2+\left(2\zeta s/\omega_n\right)+1\right]$

$$\angle\left(\frac{1}{\left(j\omega/\omega_n\right)^2+\left(2\varsigma j\omega/\omega_n\right)+1}\right)\approx-\angle 1=0^\circ \qquad \omega\ll\omega_n$$

$$\angle\left(\frac{1}{\left(j\omega/\omega_n\right)^2+\left(2\varsigma j\omega/\omega_n\right)+1}\right)=-\angle\left(2\varsigma j\right)=-90^\circ \qquad \omega=\omega_n$$

$$\angle\left(\frac{1}{\left(j\omega/\omega_n\right)^2+\left(2\varsigma j\omega/\omega_n\right)+1}\right)\approx-\angle\left(-\omega^2/\omega_n^2\right)=-180^\circ \qquad \omega\gg\omega_n$$

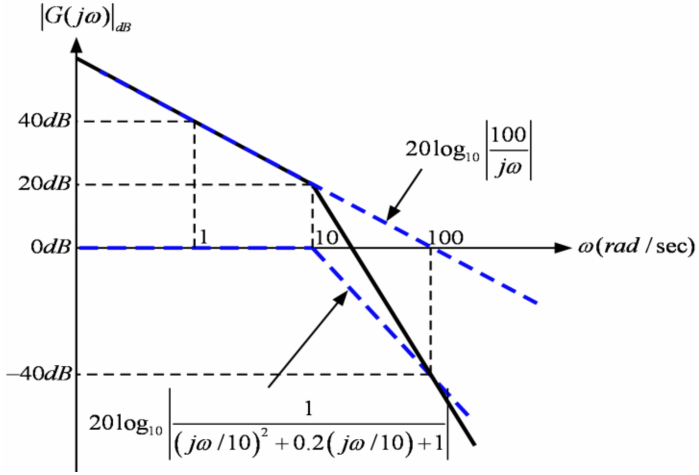


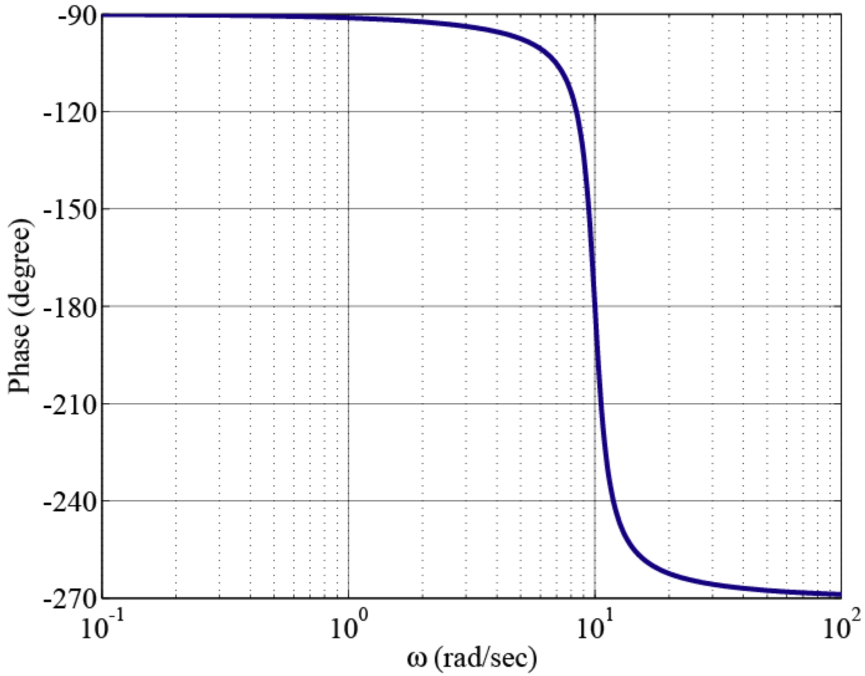


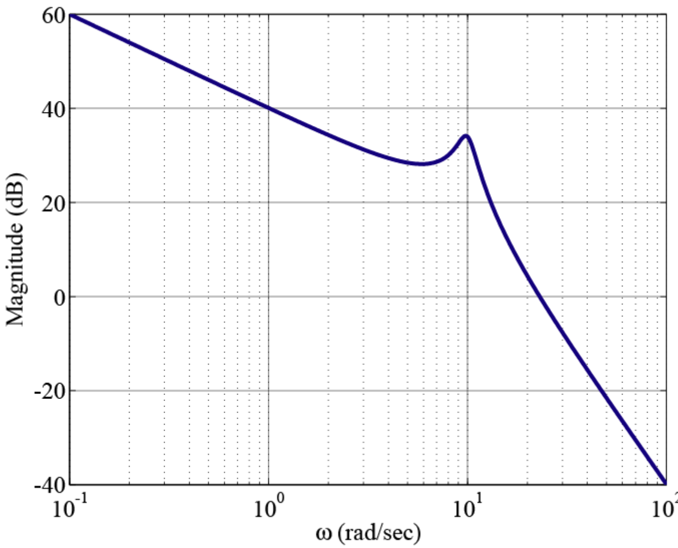
Complex conjugate poles $1/\left[(s/\omega_n)^2+(2\zeta s/\omega_n)+1\right]$

$$G(s)=\frac{10000}{s\left(s^2+2s+100\right)} \qquad G(j\omega)=\left.\frac{100}{s\left(s^2/100+2s/100+1\right)}\right|_{s=j\omega}$$

$$=\frac{100}{j\omega\left(\left(j\omega/10\right)^2+0.2\left(j\omega/10\right)+1\right)}$$



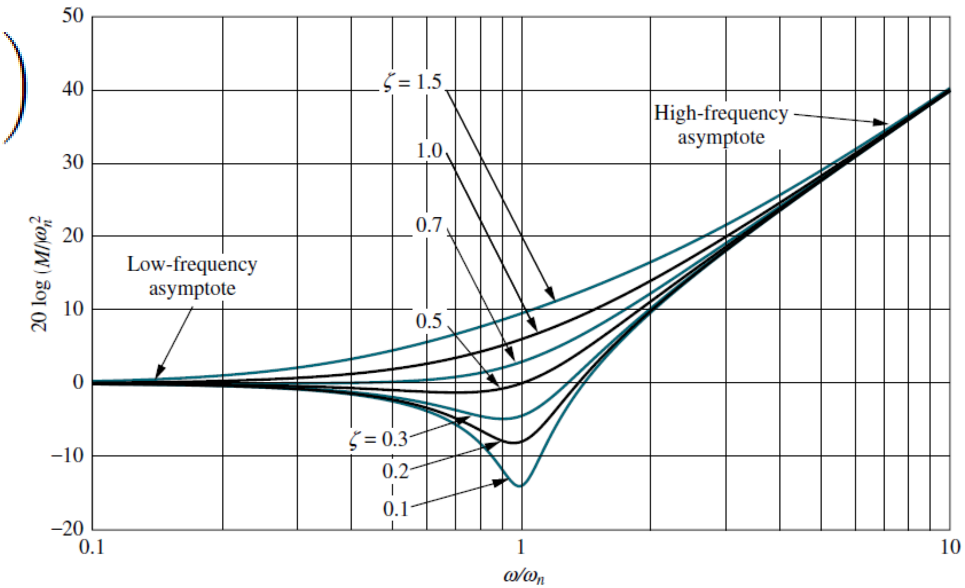




Complex conjugate zeros

$$[(s/\omega_n)^2 + (2\zeta s/\omega_n) + 1]$$

$$G(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 = \omega_n^2\left(\frac{s^2}{\omega_n^2} + 2\zeta\frac{s}{\omega_n} + 1\right)$$



At low frequencies,

$$G(s) \approx \omega_n^2 = \omega_n^2\angle 0^\circ$$

$$20\log M = 20\log|G(j\omega)| = 20\log\omega_n^2$$

At high frequencies,

$$G(s) \approx s^2$$

$$G(j\omega) \approx -\omega^2 = \omega^2\angle 180^\circ$$

$$20\log M = 20\log|G(j\omega)| = 20\log\omega^2 = 40\log\omega$$

$$M = \sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}$$

$$\text{Phase} = \tan^{-1}\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}$$