Modern Control Theory
Linear Quadratic Regulator
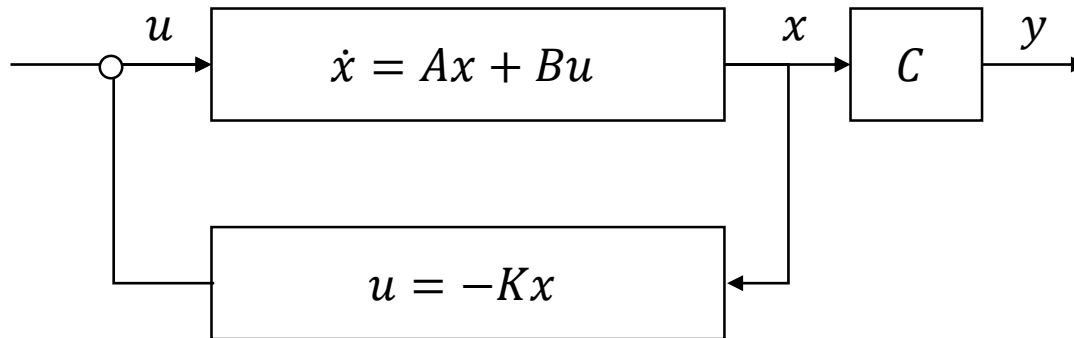
Review

State feedback configuration
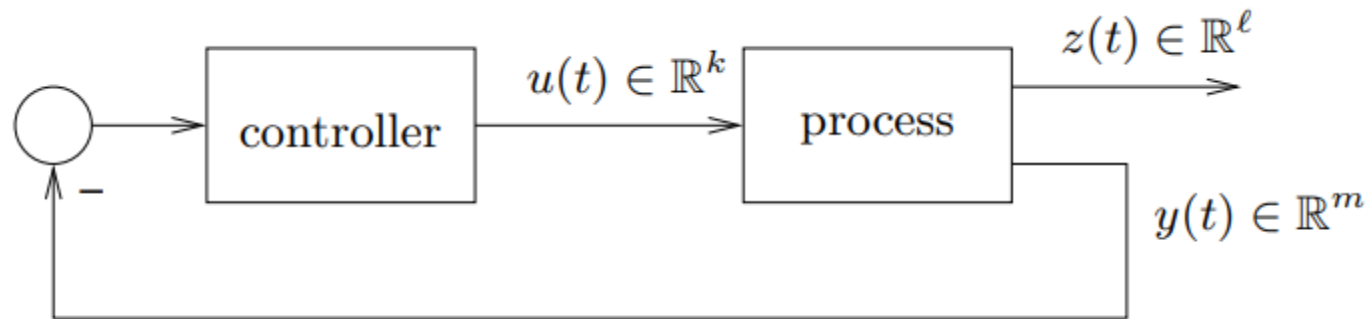
$$\dot{x} = Ax + Bu, \qquad\qquad x \in \mathbb{R}^n, \ u \in \mathbb{R}^k,$$
$$y = Cx, \qquad\qquad y \in \mathbb{R}^m,$$
$$z = Gx + Hu, \qquad\qquad z \in \mathbb{R}^\ell$$

1. The **measured output y(t)** corresponds to the signal(s) that can be measured and are therefore available for control.
2. The **controlled output z(t)** corresponds to a signal(s) that one would like to make as small as possible in the shortest possible amount of time.

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Review

Find the control input $u(t)$, $t \in [0, \infty)$ that makes the following criterion as small as possible

$$J_{\text{LQR}} := \int_0^{\infty} \|z(t)\|^2 + \rho \|u(t)\|^2 dt,$$

1. When we chose ρ very large, the most effective way to decrease J is to use little control, at the expense of a large controlled output.

2. When we chose ρ very small, the most effective way to decrease J is to obtain a very small controlled output, even if this is achieved at the expense of a large controlled output.

$$J_{\text{LQR}} := \int_0^{\infty} z(t)'\bar{Q}z(t) + \rho\, u'(t)\bar{R}u(t)\, dt,$$

where $Q \in \mathbb{R}^{\ell \times \ell}$ and $R \in \mathbb{R}^{m \times m}$ are symmetric positive-definite matrices

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General form

$$J := \int_0^\infty x(t)' Q x(t) + u'(t) R u(t) + 2x'(t) N u(t) dt.$$

① $\qquad Q = G'G, \qquad\qquad R = H'H + \rho I, \qquad\qquad N = G'H$

$$J_{\text{LQR}} := \int_0^\infty \|z(t)\|^2 + \rho \|u(t)\|^2 dt,$$

② $\qquad Q = G'\bar{Q}G, \qquad\qquad R = H'\bar{Q}H + \rho\bar{R}, \qquad\qquad N = G'\bar{Q}H.$

$$J_{\text{LQR}} := \int_0^\infty z(t)'\bar{Q}z(t) + \rho\, u'(t)\bar{R}u(t)\, dt,$$

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Lemma

Let P be a symmetric matrix. For every control input
$u(t)$, $t \in [0, \infty)$ for which $x(t) \to 0$ as $t \to \infty$,

$$\int_0^\infty x'(A'P + PA)x + 2x'PBu \, dt = -x(0)'Px(0).$$

Proof of Lemma 1.

$$\int_0^\infty x'(A'P + PA)x + 2x'PBu \, dt = \int_0^\infty (x'A' + u'B')Px + x'P(Ax + Bu) \, dt$$

$$= \int_0^\infty \dot{x}'Px + x'P\dot{x} \, dt = \int_0^\infty \frac{\mathrm{d}(x'Px)}{\mathrm{d}t} dt = \lim_{t \to \infty} x'(t)'Px(t) - x(0)'Px(0).$$

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Square completion

$$J := \int_0^\infty x(t)'Qx(t) + u'(t)Ru(t) + 2x'(t)Nu(t)dt.$$

$$x(0)'P\,x(0)$$

$$J_{\text{LQR}} = x(0)'P\,x(0) + \int_0^\infty x'(A'P + PA + Q)x + u'Ru + 2x'(PB + N)u\ dt.$$

$$u'Ru + 2x'(PB + N)u = (u - u_0)'R(u - u_0) - x'(PB + N)R^{-1}(B'P + N')x$$

where

$$u_0 := -R^{-1}(B'P + N')x.$$

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Square completion

$$J_{\text{LQR}} = x(0)'P\,x(0) + \int_0^\infty x'(A'P + PA + Q)x + u'Ru + 2x'(PB + N)u\,dt.$$

$$u_0 := -R^{-1}(B'P + N')x.$$

$$J_{\text{LQR}} = J_0 + \int_0^\infty (u - u_0)'R(u - u_0)\,dt,$$

where

$$J_0 := x(0)'P\,x(0) + \int_0^\infty x'\big(A'P + PA + Q - (PB + N)R^{-1}(B'P + N')\big)x\,dt.$$

$$A'P + PA + Q - (PB + N)R^{-1}(B'P + N') = 0.$$

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Theorem

Assume that there exists a symmetric solution P to the Algebraic Riccati Equation for which $A - BR^{-1}(B'P + N')$ is Hurwitz. Then the feedback law
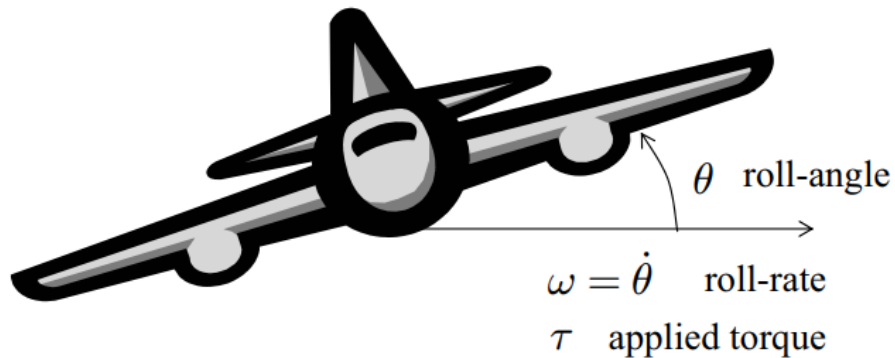
$$u(t) := -Kx(t), \quad \forall t \geq 0, \qquad K := R^{-1}(B'P + N'),$$

minimizes the LQR criterion (J-LQR) and leads to

$$J_{\mathrm{LQR}} := \int_0^\infty x'Qx + u'Ru + 2x'Nu \, dt = x'(0)Px(0).$$

Example

$$x := \begin{bmatrix} \theta & \omega & \tau \end{bmatrix}'$$

θ roll-angle

$\omega = \dot{\theta}$ roll-rate

τ applied torque

$\dot{\theta} = \omega$

$\dot{\omega} = -.875\omega - 20\tau$

$\dot{\tau} = -50\tau + 50u$