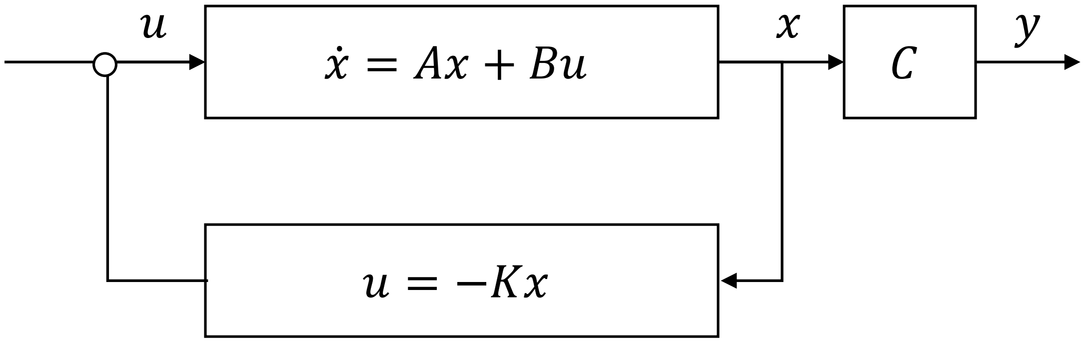
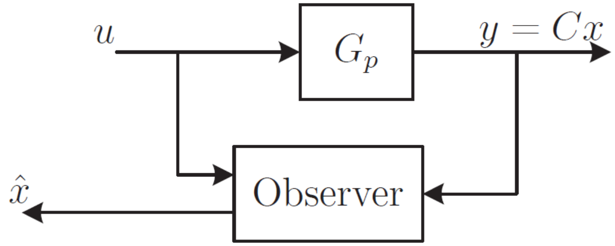
Modern Control Theory

Kalman Filter



State observer

State Observer:



$$G_p : \begin{cases} x(k+1) = \Phi x(k) + \Gamma u(k) \\ y(k) = Cx(k) \end{cases}$$

State observer

$$\hat{x}(k+1) = \Phi \hat{x}(k) + \Gamma u(k) + L(y(k) - C\hat{x}(k))$$

Estimation error $e(k) = x(k) - \hat{x}(k)$

$$e(k+1) = \Phi e(k) - LCe(k) = (\Phi - LC)e(k)$$ state observation error dynamics

The observer gain L can be chosen such that $e \rightarrow 0$, irrespective of u (provided it's known and used)

State observer

Prediction-Correction State Observer:

An LTI system

$$G_p : \begin{cases} x(k+1) = \Phi x(k) + \Gamma u(k) \\ y(k) = Cx(k) \end{cases}$$

Prediction:

$$\bar{x}(k+1) = \Phi \hat{x}(k) + \Gamma u(k)$$

Correction:

$$\hat{x}(k) = \bar{x}(k) + L(y(k) - C\bar{x}(k))$$

Estimation error $\bar{e}(k) = x(k) - \bar{x}(k)$

$$\bar{e}(k+1) = (\Phi - \Phi LC)\bar{e}(k)$$ state observation error dynamics

Note: $\bar{e}(\cdot)$ convergent $\Rightarrow$ $e(\cdot) = x(\cdot) - \hat{x}(\cdot)$ convergent.

Kalman filter: white Gaussian noise

Discrete Kalman Filter

Let us consider a discrete-time system

$$G_p : \begin{cases} x_{k+1} = \Phi_k x_k + w_k \\ y_k = C_k x_k + v_k \end{cases}$$

where

w_k : white sequence with known covariance, $\sim N(0, Q_k)$

v_k : white sequence measurement error with known covariance, $\sim N(0, R_k)$

The covariance matrices for the w_k and v_k

$$\mathbb{E}[w_k w_k^T] = Q_k, \;\; \mathbb{E}[w_k w_j^T] = 0 \;\; (j \neq k)$$

$$\mathbb{E}[v_k v_k^T] = R_k, \;\; \mathbb{E}[v_k v_j^T] = 0 \;\; (j \neq k)$$

$$\mathbb{E}[w_k v_j^T] = 0 \;\; (\forall k,\; j)$$

Kalman filter

Prediction (*a priori*) estimate $\bar{x}_k$

Prediction (*a priori*) estimation error

$$\bar{e}_k = x_k - \bar{x}_k$$

Prediction (*a priori*) error covariance matrix

$$\bar{\Sigma}_k = \mathbb{E}[\bar{e}_k \bar{e}_k^T] = \mathbb{E}[(x_k - \bar{x}_k)(x_k - \bar{x}_k)^T]$$

How to use the measurement y_k to improve the prior estimate $\bar{x}_k$

Kalman filter

We choose

$$\hat{x}_k = \bar{x}_k + L_k(y_k - C_k\bar{x}_k)$$

where

$\hat{x}_k$: the updated (*a posteriori*) estimate

L_k : a gain to be determined

How to find the gain L_k that yields an updated estimate that is optimal in some sense

- minimum mean-square error as a performance criterion

Kalman filter

Updated (*a posteriori*) estimation error:

$$e_k = x_k - \hat{x}_k$$

The covariance associated with the updated estimate error:

$$\Sigma_k = \mathbb{E}[e_k e_k^T] = \mathbb{E}[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T]$$

Using

$$\hat{x}_k = \bar{x}_k + L_k(y_k - C\bar{x}_k) = (I - L_k C)\bar{x}_k + L_k C x_k + L_k v_k$$

$$e_k = x_k - \hat{x}_k = (I - L_k C)\bar{e}_k - L_k v_k$$

we have

$$\Sigma_k = \mathbb{E}[e_k e_k^T] = (I - L_k C)\bar{\Sigma}_k (I - L_k C)^T + L_k R_k L_k^T$$

(The *a priori* estimation error $\bar{e}_k$ uncorrelated with v_k, $\mathbb{E}[\bar{e}_k v_k^T] = 0$)

Kalman filter: optimization

Optimization:

Need to solve an optimization problem

$$\min_{L_k} \mathrm{tr}[\Sigma_k]$$

subject to Convergence

where

$$\begin{aligned}\Sigma_k = \mathbb{E}[e_k e_k^T] &= (I - L_k C_k)\bar{\Sigma}_k (I - L_k C_k)^T + L_k R_k L_k^T \\ &= \bar{\Sigma}_k - L_k C_k \bar{\Sigma}_k - \bar{\Sigma}_k C_k^T L_k^T + L_k (C_k \bar{\Sigma}_k C_k^T + R_k) L_k^T\end{aligned}$$

Good to note:

For $A = [a_{kl}] \in \mathbb{C}^{n\times m}$, $B = [b_{kl}] \in \mathbb{C}^{n\times m}$,

$$\mathrm{tr}\left[A^T B\right] = \sum_{k=1}^{n}\sum_{l=1}^{m} a_{kl} b_{kl} = a_{11} b_{11} + \cdots + a_{nm} b_{nm}$$

If $A = B$,

$$\mathrm{tr}\left[A^T A\right] = \sum_{k=1}^{n}\sum_{l=1}^{m} a_{kl}^2 = a_{11}^2 + \cdots + a_{nm}^2$$

Kalman filter: optimization

$$\frac{d\mathrm{tr}[\Sigma_k]}{dL_k} = -2(C_k\bar{\Sigma}_k)^T + 2L_k(C_k\bar{\Sigma}_k C_k^T + R_k) = 0$$

Optimal gain

$$\boxed{L_k = \bar{\Sigma}_k C_k^T (C_k\bar{\Sigma}_k C_k^T + R_k)^{-1} \qquad \text{(Kalman gain)}}$$

The covariance matrix associated with the optimal estimate

$$\Sigma_k = (I - L_k C_k)\bar{\Sigma}_k (I - L_k C_k)^T + L_k R_k L_k^T$$

Substituting the optimal gain leads to

$$\boxed{\Sigma_k = (I - L_k C_k)\bar{\Sigma}_k}$$

Kalman filter

Prediction model

$$\bar{x}_{k+1} = \Phi_k \hat{x}_k$$

The error covariance matrix associated with $\bar{x}_{k+1}$

$$\bar{e}_{k+1} = x_{k+1} - \bar{x}_{k+1} = \Phi_k x_k + w_k - \Phi_k \hat{x}_k = \Phi_k e_k + w_k$$

The prediction error covariance matrix $\bar{\Sigma}_{k+1} = \mathbb{E}[\bar{e}_{k+1}\bar{e}_{k+1}^T]$

$$\boxed{\bar{\Sigma}_{k+1} = \Phi_k \Sigma_k \Phi_k^T + Q_k}$$

($\mathbb{E}[e_k w_k^T] = 0,\ e_k$ uncorrelated with w_k)

Kalman filter

Convergence?

Assume stationary, fixed sample period process

$$\bar{\Sigma}_{k+1} = \Phi\Sigma_k\Phi^T + Q_k = (\Phi - \Phi L_k C)\bar{\Sigma}_k(\Phi - \Phi L_k C)^T + \Phi L_k R_k L_k^T \Phi^T + Q_k$$

assures

$$(\Phi - \Phi L_k C)\bar{\Sigma}_k(\Phi - \Phi L_k C)^T - \bar{\Sigma}_{k+1} + Q_k < 0$$

Note:

- Steady state prediction error dynamics
- $\bar{\Sigma} > 0$ for the Lyapunov inequality

Kalman filter: summary

Discrete Kalman Filter Algorithm:

- Correction update (using measurement y_k):

$$L_k = \bar{\Sigma}_k C_k^T (C_k \bar{\Sigma}_k C_k^T + R_k)^{-1}$$

$$\hat{x}_k = \bar{x}_k + L_k(y_k - C_k \bar{x}_k)$$

$$\Sigma_k = (I - L_k C_k)\bar{\Sigma}_k$$

- Prediction update:

$$\bar{x}_{k+1} = \Phi_k \hat{x}_k$$

$$\bar{\Sigma}_{k+1} = \Phi_k \Sigma_k \Phi_k^T + Q_k$$

Kalman filter: summary

Time invariant system: steady state soln.

Using

$$\Sigma_k = (I - LC)\bar{\Sigma}_k$$

and

$$\bar{\Sigma}_{k+1} = \Phi \Sigma_k \Phi^T + Q$$

we obtain

$$\bar{\Sigma}_{k+1} = \Phi (I - LC)\bar{\Sigma}_k \Phi^T + Q$$

Let $Y = \bar{\Sigma}_\infty$, then

$$\boxed{L = YC^T(CYC^T + R)^{-1}}$$

$$Y = \Phi(I - LC)Y\Phi^T + Q$$

Substituting $I - LC = I - YC^T(CYC^T + R)^{-1}C$ leads to

$$\boxed{Y = \Phi Y \Phi^T - \Phi Y C^T(CYC^T + R)^{-1}CY\Phi^T + Q}$$

Extended Kalman filter

Extended Kalman Filter

Most realistic robotic problems involve nonlinear functions

$$\begin{cases} x_{k+1} = f_k(x_k, u_{k+1}) + w_k \\ z_k = h_k(x_k) + v_k \end{cases}$$

The covariance matrices for w_k and v_k

$$\mathbb{E}[w_k w_k^T] = Q_k,\ \ \mathbb{E}[w_k w_j^T] = 0\ \ (j \neq k)$$

$$\mathbb{E}[v_k v_k^T] = R_k,\ \ \mathbb{E}[v_k v_j^T] = 0\ \ (j \neq k)$$

$$\mathbb{E}[w_k v_j^T] = 0\ \ (\forall k,\ j)$$

Extended Kalman filter

Kalman Filter Revisited:

- Correction update (using measurement z_k):

$$L_k = \bar{\Sigma}_k C_k^T (C_k \bar{\Sigma}_k C_k^T + R_k)^{-1} \qquad \text{(Kalman gain)}$$

$$\hat{x}_k = \bar{x}_k + L_k (z_k - C_k \bar{x}_k)$$

$$\Sigma_k = (I - L_k C_k) \bar{\Sigma}_k$$

- Prediction update:

$$\bar{x}_{k+1} = \Phi_k \hat{x}_k$$

$$\bar{\Sigma}_{k+1} = \Phi_k \Sigma_k \Phi_k^T + Q_k$$

Extended Kalman filter

EKF Linearization: First Order Taylor Series Expansion

- Prediction:

$$
\begin{aligned}
f(x_{k-1}, u_k) &\approx f(\hat{x}_{k-1}, u_k) + \frac{\partial f(x_{k-1}, u_k)}{\partial x_{k-1}}\bigg|_{\hat{x}_{k-1}} (x_{k-1} - \hat{x}_{k-1}) \\
&= f(\hat{x}_{k-1}, u_k) + \Phi_k (x_{k-1} - \hat{x}_{k-1})
\end{aligned}
$$

- Correction:

$$
\begin{aligned}
h(x_k) &\approx h(\bar{x}_k) + \frac{\partial h(x_k)}{\partial x_k}\bigg|_{\bar{x}_k} (x_k - \bar{x}_k) \\
&= h(\bar{x}_k) + C_k (x_k - \bar{x}_k)
\end{aligned}
$$

Extended Kalman filter

Extended Kalman Filter Algorithm:

- Correction update (using measurement z_k):

$$L_k = \bar{\Sigma}_k C_k^T (C_k \bar{\Sigma}_k C_k^T + R_k)^{-1}$$

$$\hat{x}_k = \bar{x}_k + L_k(z_k - h_k(\bar{x}_k))$$

$$\Sigma_k = (I - L_k C_k)\bar{\Sigma}_k$$

- Prediction update:

$$\bar{x}_{k+1} = f_k(\hat{x}_k, u_{k+1})$$

$$\bar{\Sigma}_{k+1} = \Phi_k \Sigma_k \Phi_k^T + Q_k$$

$$\Phi_k = \left.\frac{\partial f(x_{k-1}, u_k)}{\partial x_{k-1}}\right|_{\hat{x}_{k-1}}, \quad C_k = \left.\frac{\partial h(x_k)}{\partial x_k}\right|_{\bar{x}_k}$$

Summary

Generalized plant

$$G = \left[\begin{array}{c|cc} A & B_w & B \\ \hline C_{zx} & 0 & D_{zu} \\ C & D_w & 0 \end{array}\right] : \begin{cases} \dot{x} = Ax + B_w w + Bu \\ z = C_{zx}x + D_{zu}u \\ y = Cx + D_w w \end{cases}$$

Controller

$$\left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] : \begin{cases} \dot{\hat{x}} = A_c\hat{x} + B_c y \\ u = C_c\hat{x} \end{cases}$$

- Kalman filter:

$$\hat{x} = A\hat{x} + Bu - L(C\hat{x} - y)$$

- State feedback controller

$$u = -F\hat{x}$$

Summary

Assumptions on the system matrix:

A1) (A, B) stabilizable

A2) $D_{zu}^T D_{zu}$ invertible

A3) $D_{zu}^T C_{zx} = 0$

A4) (C_{zx}, A) has no unobservable modes on the imaginary axis

B1) (C, A) detectable

B2) $D_w D_w^T$ invertible

B3) $D_w B_w^T = 0$

B4) (A, B_w) has no uncontrollable modes on the imaginary axis

Summary

Closed loop sys

$$\tilde{G}_{zw} = \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & 0 \end{array}\right] \quad : \quad \begin{cases} \begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B_w \\ B_cD_w \end{bmatrix} w \\ z = \begin{bmatrix} C_{zx} & D_{zu}C_c \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} \end{cases}$$

Let

$$\tilde{A} = \begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_w \\ B_cD_w \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} C_{zx} & D_wC_c \end{bmatrix}$$

and

$$R_x = C_{zx}^T C_{zx}, \;\; R_u = D_{zu}^T D_{zu}$$

$$V_x = B_w B_w^T, \;\; V_y = D_w D_w^T$$

Summary

Optimization problem: Find (A_c, B_c, C_c) s.t. $\tilde{A}$ is asym. stable and $\|\tilde{T_{zw}}\|_2^2$ is minimized.

Optimal controller

$$-B_c = -L = -YC^T V_y^{-1}$$: Kalman filter gain matrix

$$C_c = -F = -R_u^{-1} B^T X$$: Optimal state feedback matrix

$$A_c = A + BC_c - B_c C = A - BF - LC$$

where X and Y

$$A^T X + XA + R_x - XBR_u^{-1} B^T X = 0$$

$$AY + YA^T + V_x - YC^T V_y^{-1} CY = 0$$