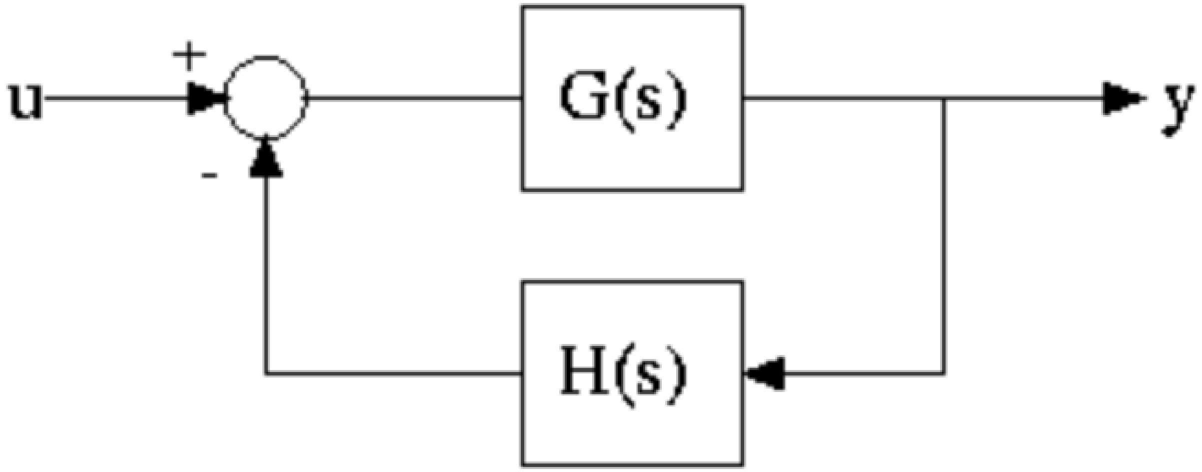
System Control

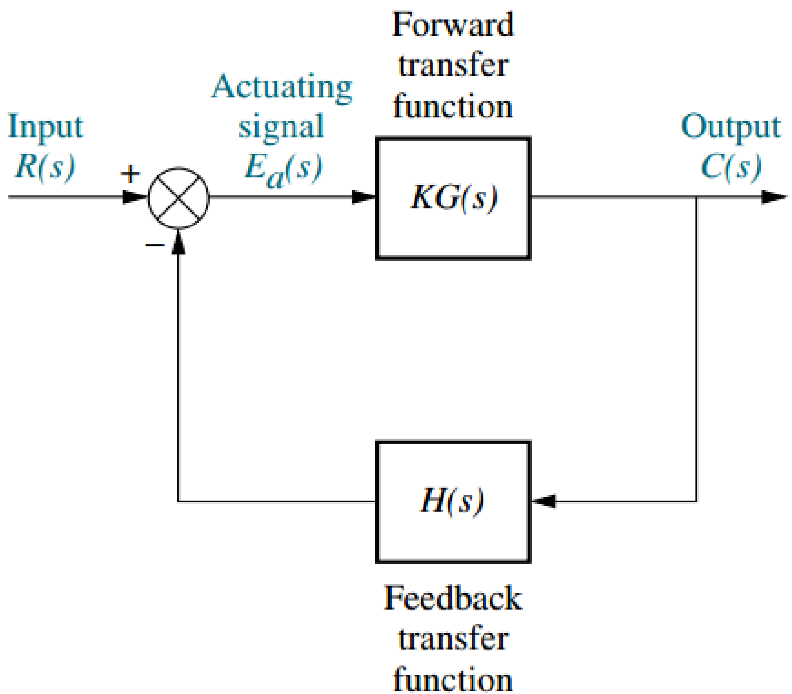
Root locus techniques

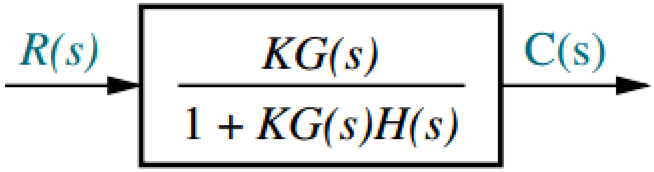


Introduction

Root locus?

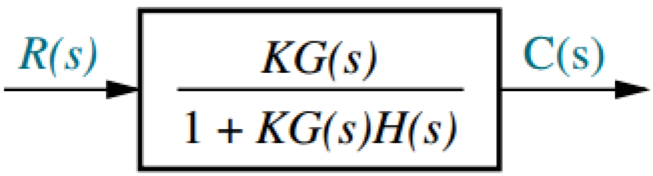
The root locus can be used to describe qualitatively the performance of a system as various parameters are changed.





Introduction

Root locus?



$$G(s) = \frac{N_G(s)}{D_G(s)} \qquad H(s) = \frac{N_H(s)}{D_H(s)}$$

$$T(s) = \frac{K N_G(s) D_H(s)}{D_G(s) D_H(s) + K N_G(s) N_H(s)}$$

Vector representation of complex numbers

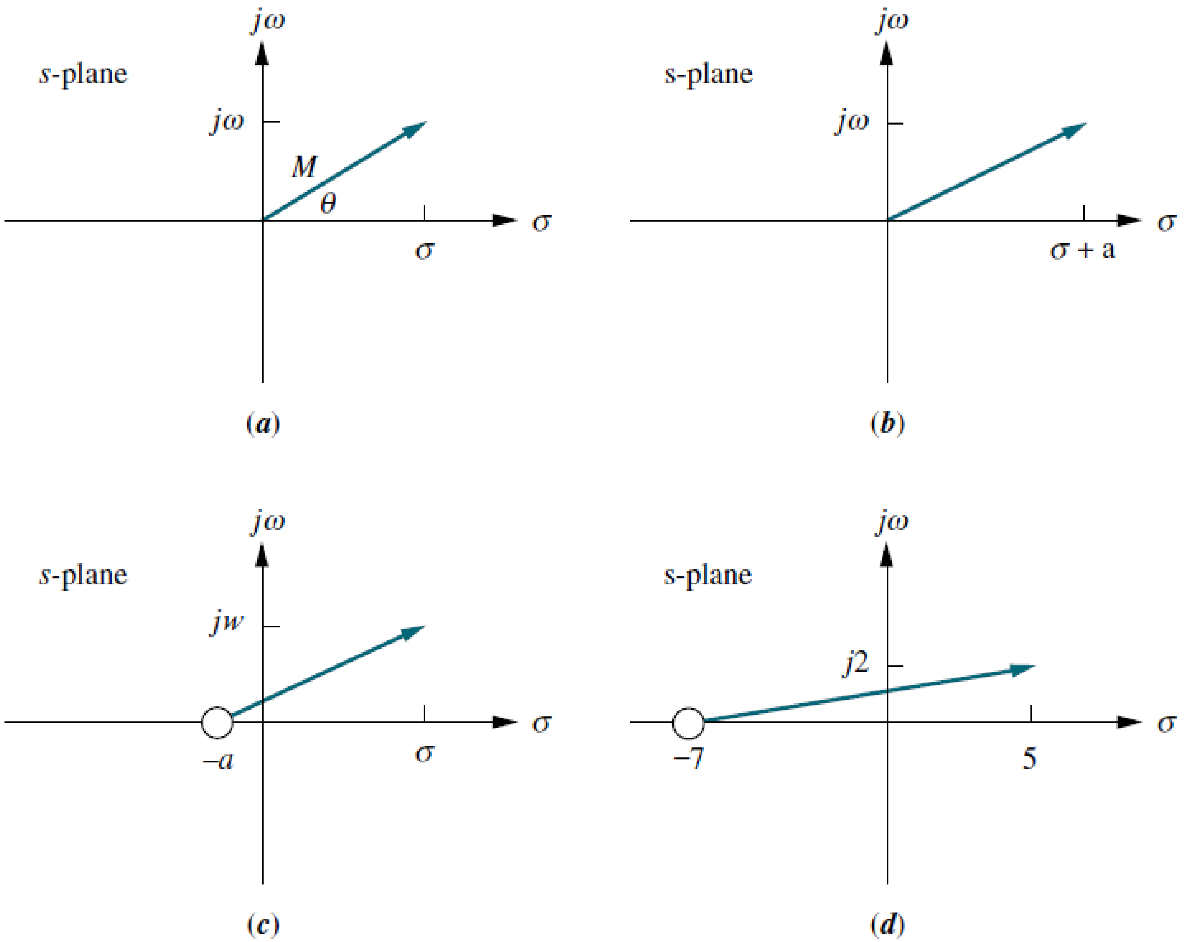


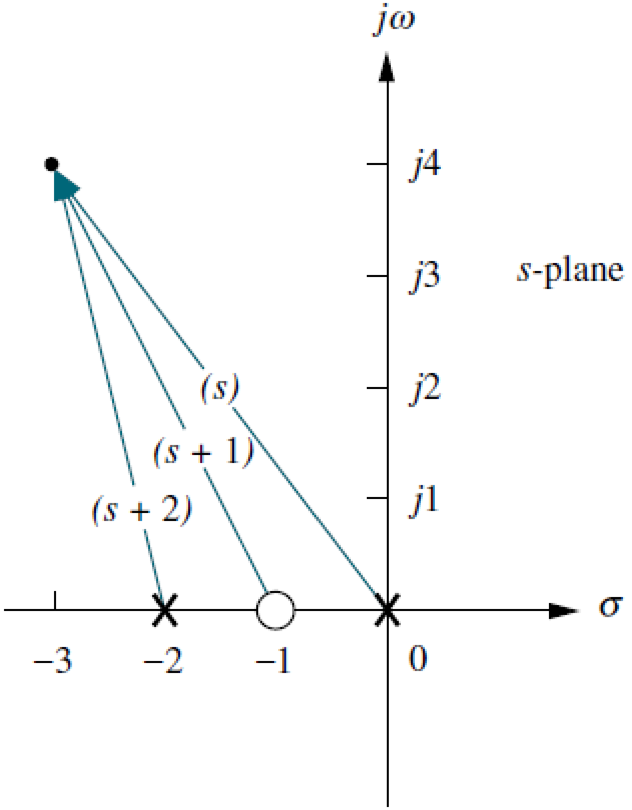
FIGURE 8.2 Vector representation of complex numbers: **a.** $s = \sigma + j\omega$; **b.** $(s + a)$; **c.** alternate representation of $(s + a)$; **d.** $(s + 7)|_{s \to 5+j2}$

Vector representation of complex numbers

PROBLEM: Given

$$F(s) = \frac{(s+1)}{s(s+2)}$$

find $F(s)$ at the point $s = -3 + j4$.



The vector originating at the zero at −1 is

$$\sqrt{20}\angle 116.6^\circ$$

The vector originating at the pole at the origin is

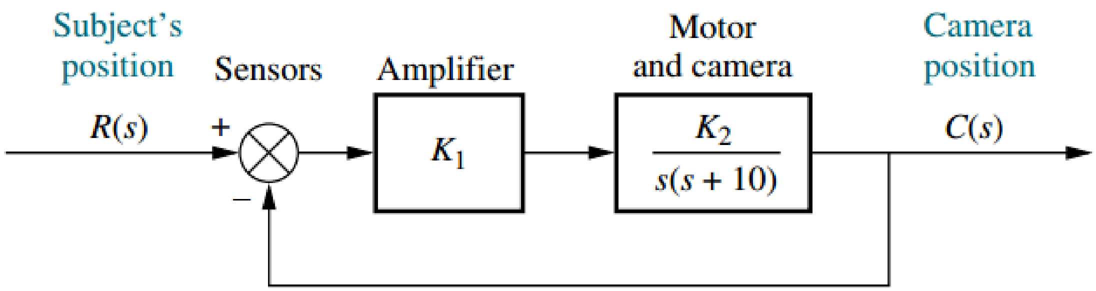
$$5\angle 126.9^\circ$$

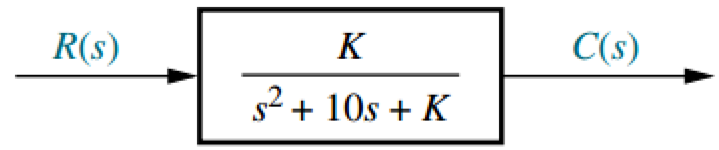
The vector originating at the pole at −2 is

$$\sqrt{17}\angle 104.0^\circ$$

$$M\angle\theta = \frac{\sqrt{20}}{5\sqrt{17}}\angle 116.6^\circ - 126.9^\circ - 104.0^\circ = 0.217\angle -114.3^\circ$$

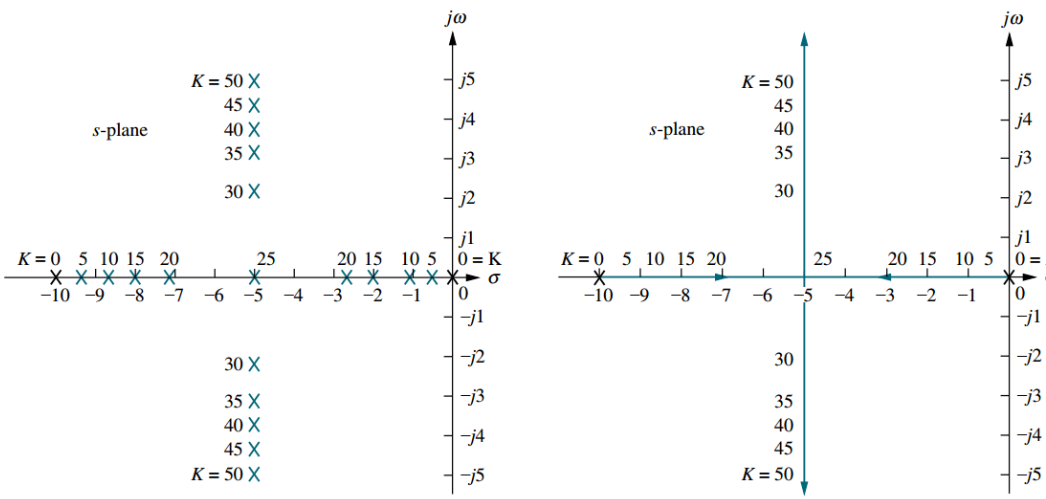
Defining the root locus





where $K = K_1K_2$

K	Pole 1	Pole 2
0	–10	0
5	–9.47	–0.53
10	–8.87	–1.13
15	–8.16	–1.84
20	–7.24	–2.76
25	–5	–5
30	$-5 + j2.24$	$-5 - j2.24$
35	$-5 + j3.16$	$-5 - j3.16$
40	$-5 + j3.87$	$-5 - j3.87$
45	$-5 + j4.47$	$-5 - j4.47$
50	$-5 + j5$	$-5 - j5$



Properties of the root locus

Closed-loop transfer function

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

The characteristic polynomial in the denominator becomes zero

$$KG(s)H(s) = -1 = 1\angle(2k+1)180^\circ \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots$$

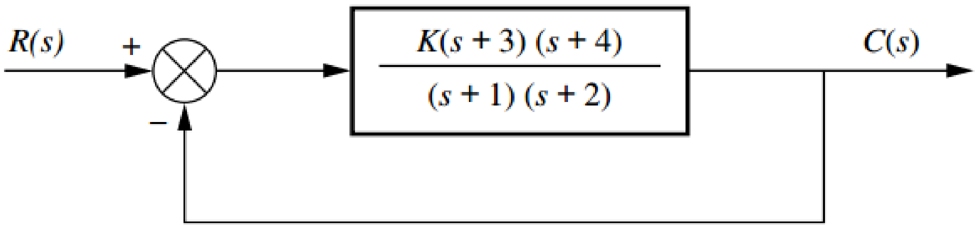
or

$$|KG(s)H(s)| = 1$$

and

$$\angle KG(s)H(s) = (2k+1)180^\circ$$

Properties of the root locus



$$|KG(s)H(s)| = 1$$

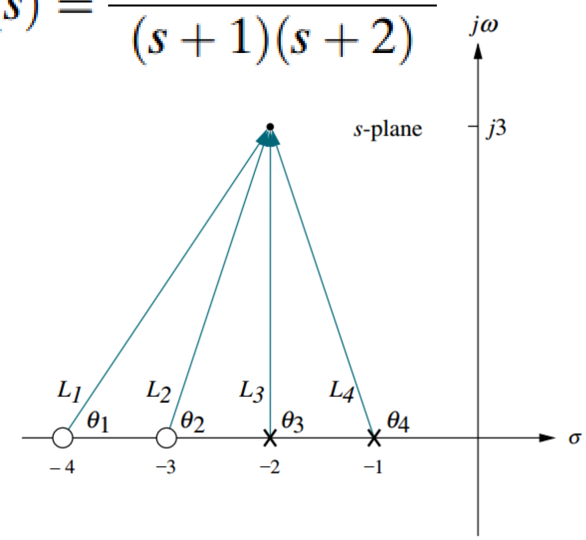
$$\angle KG(s)H(s) = (2k+1)180^\circ$$

Open-loop TF

$$KG(s)H(s) = \frac{K(s+3)(s+4)}{(s+1)(s+2)}$$

Closed-loop TF

$$T(s) = \frac{K(s+3)(s+4)}{(1+K)s^2 + (3+7K)s + (2+12K)}$$



Properties of the root locus

$$KG(s)H(s) = \frac{K(s+3)(s+4)}{(s+1)(s+2)}$$

$$-2+j3 \quad \text{vs} \quad -2+j(\sqrt{2}/2)$$

$$M = \frac{\prod \text{zero lengths}}{\prod \text{pole lengths}} = \frac{\prod_{i=1}^{m} |(s+z_i)|}{\prod_{j=1}^{n} |(s+p_j)|}$$

$$\theta = \sum \text{zero angles} - \sum \text{pole angles} = \sum_{i=1}^{m} \angle(s+z_i) - \sum_{j=1}^{n} \angle(s+p_j)$$

$$-2+j3$$

$$\theta_1 + \theta_2 - \theta_3 - \theta_4 = 56.31^\circ + 71.57^\circ - 90^\circ - 108.43^\circ = -70.55^\circ$$

$$-2+j(\sqrt{2}/2)$$

$$\theta_1 + \theta_2 - \theta_3 - \theta_4 = 180^\circ$$

$$K = \frac{L_3 L_4}{L_1 L_2} = \frac{\frac{\sqrt{2}}{2}(1.22)}{(2.12)(1.22)} = 0.33$$

Sketching the root locus

1. **Number of branches.**

   The number of branches of the root locus equals the number of closed-loop poles.

2. **Symmetry.**

   The root locus is symmetrical about the real axis.

3. **Real-axis segments.**

   On the real axis, for $K > 0$ the root locus exists to the left of an odd number of real-axis, finite open-loop poles and/or finite open-loop zeros.

4. **Starting and ending points.**

   The root locus begins at the finite and infinite poles of $G(s)H(s)$ and ends at the finite and infinite zeros of $G(s)H(s)$.

Sketching the root locus

5. Behavior at infinity.

The root locus approaches straight lines as asymptotes as the locus approaches infinity. Further, the equation of the asymptotes is given by the real-axis intercept, σ_a *and angle,* θ_a *as follows:*

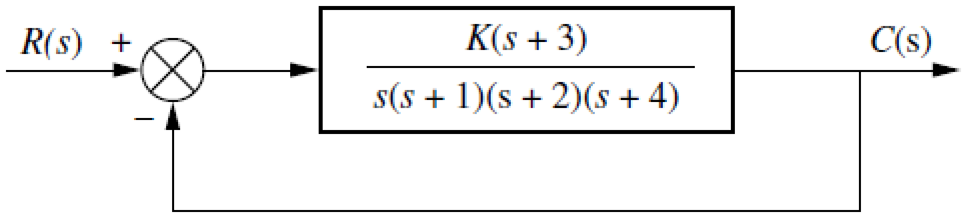
$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}}$$

$$\theta_a = \frac{(2k+1)\pi}{\#\text{finite poles} - \#\text{finite zeros}}$$

where $k = 0, \pm 1, \pm 2, \pm 3$ *and the angle is given in radians with respect to the positive extension of the real axis.*

Sketching the root locus

PROBLEM: Sketch the root locus for the system



$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}}$$

$$\theta_a = \frac{(2k+1)\pi}{\#\text{finite poles} - \#\text{finite zeros}}$$

$$\sigma_a = \frac{(-1-2-4)-(-3)}{4-1} = -\frac{4}{3}$$

$$\begin{aligned}\theta_a &= \frac{(2k+1)\pi}{\#\text{finite poles} - \#\text{finite zeros}}\\ &= \pi/3 \qquad \text{for } k=0\\ &= \pi \qquad \text{for } k=1\\ &= 5\pi/3 \qquad \text{for } k=2\end{aligned}$$

Sketching the root locus